Hamilton's Discovery of the Quaternions and Creativity in Mathematics

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Creativity is, arguably, mankind's—let alone a mathematician's— most important, and valuable, attribute. Yet it remains tantalisingly elusive. What, precisely, is it? Is it a 'natural gift'? Or is it a skill that can be learnt? Hamilton's discovery of quaternions is well-known as an example of a 'sudden flash of inspiration', but this is not the full story, for he had been thinking about the problem for a long time before the solution emerged. But what had he been thinking about? Fortunately, Hamilton left some important clues—clues that can be built upon to identify a process of creative discovery that can be generalized very broadly.

Sir William Rowan Hamilton's discovery of the quaternions is a famous example of a flash of inspiration. To quote his own words, as he was walking with his wife along the Royal Canal in Dublin on 16 October 1843, "an *electric* circuit seemed to *close*; and a spark flashed forth" [8], that spark being the quaternion $a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d$, with the new items j and k defined such that $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$.

So excited was he that he carved these formulas on a stone of Brougham Bridge. And on the following day, Hamilton wrote a letter to John T. Graves [7], in which he offers some rare insights into the process of mathematical discovery, showing that although the moment at which everything fell into place was so sudden, the discovery of quaternions was by no means an instantaneous "spark flashing forth", but the result of much hard work, experimentation, and deep thinking.

Hamilton's end-result, in modern terminology, was the construction of a normed division algebra [4]. In Hamilton's day, the known dimensions of such algebras were 1 (real numbers) and 2 (complex numbers), and in his letter to Graves, Hamilton talks of having "long wished ... to possess a Theory of Triplets, analogous to my published Theory of Couplets". This clearly indicates the Hamilton had for some time been thinking about how to generalize the concept of complex numbers, for which the most obvious idea is to extend "couplets" $a + \mathrm{i}b$ into "triplets" $a + \mathrm{i}b + \mathrm{j}c$ by introducing j as an analogue to i.

When, however, Hamilton multiplied two triplets, $a + \mathrm{i}b + \mathrm{j}c$ and $x + \mathrm{i}y + \mathrm{j}z$, he ran into difficulty with the product ij: to quote Hamilton once more, "But what are we to do with ij? ... This might tempt us to take $\mathrm{ij} = 1$ or $\mathrm{ij} = -1$; but with neither assumption shall we have the sum of the squares of the coefficients of 1, i and j = to the product of the corresponding sums of squares in the factors ... Behold me therefore tempted for a moment to fancy that $\mathrm{ij} = 0$. But this seemed odd and uncomfortable ... [so I assumed] what seemed to me to be less harsh, namely that $\mathrm{ij} = -\mathrm{ji}$. I made therefore $\mathrm{ij} = \mathrm{k}$, $\mathrm{ji} = -\mathrm{k}$."

Hamilton had not only discovered that i and j are non-commutative but had also stumbled on the possibility of an additional imaginary quantity k, as he described in the delightful sentence "And here there dawned on me the notion that we must admit, in some sense, of a *fourth dimension* ... or transferring the paradox to algebra, must admit a *third* distinct imaginary symbol k, not to be confounded with either i or j, but equal to the product of the first as multiplier, and the second as multiplicand; and therefore was led to introduce *quaternions*, such as $a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d$."

Hamilton's letter, and subsequent paper [6], present many fascinating personal details relating to his discovery, but three fundamental features stand out.

The first is curiosity, a desire to enquire—and, in Hamilton's case, with the luxury of having the space to enquire as an abstract exercise, rather than as driven by the urgency of a problem that demanded solution. Real numbers and complex numbers were well known and understood, and served their purpose well. No one was saying, "Hey, Hamilton! Complex numbers don't do [whatever]! Fix it, will you?". Rather, Hamilton was curious as to whether a "Theory of Triplets", analogous to the "Theory of Couplets" might exist, and what it might look like.

That curiosity led to the second fundamental feature, exploration, a well-directed search, in which he examined the algebra of triplets $a + \mathrm{i}b + \mathrm{j}c$, and discovered—no doubt to his initial disappointment and frustration—that things didn't work out, especially as regards the product ij.

And in considering different alternative possibilities, Hamilton discovered that if $\mathrm{ij} \neq \mathrm{ji}$, but rather that $\mathbf{ij} = -\mathbf{ji}$, then everything worked. That must have been truly startling, for in 1843, non-commutative multiplication 'broke all the rules'. The non-commutative properties of matrix multiplication were not to be described until the work of Arthur Cayley some 15 years later [2], and although it is possible that Hamilton was aware of non-commutation in the context of rotations, the commutative properties of numbers and algebraic expressions was so ingrained into everyone's knowledge that the very possibility that the product $\mathbf{ij}$ might be non-commutative would, to most people, have been unthinkable—or, if thought, at once dismissed as 'ridiculous'. But Hamilton both thought it and developed it, and in so doing had been willing to throw away the conventional wisdom about multiplication. Which makes 'unlearning' the third fundamental feature of the story. And as Hamilton describes, the recognition that $\mathbf{i}$ could not commute with $\mathbf{j}$ led him to introduce $\mathbf{k}$ as $\mathbf{ij} = \mathbf{k}$, and that triggered the possibility of, as he put it, a "fourth dimension", leading him to explore the entity $a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$, at which point everything worked.

Those three features—curiosity, exploration, and a willingness to unlearn—are exemplified by Hamilton's discovery of quaternions, but we propose here that they are in fact fundamental features of all creativity. Furthermore, they are unified when creativity is perceived not so much as the quest for the *new*, but as the discovery of the *different*.

That might appear to be both strange and counterintuitive. Yes, in 1843, quaternions were 'new'; more importantly, however, they were different from their immediate antecedents, real numbers and complex numbers, but different only in two respects: four terms rather than one or two and, crucially, employing non-commutative multiplication. Furthermore, in discovering quaternions, Hamilton did not just 'stare into blue space hoping that the lightning would strike'; on the contrary, he started from what he knew, the algebra of real and complex numbers, and undertook a thorough and diligent exploration of possibilities.

When seeking to be creative, the value of starting from what you already know is highlighted by this insight from Arthur Koestler's book The Act of Creation [11]: "The creative act is not an act of creation in the sense of the Old Testament. It does not create something out of nothing; it uncovers, selects, re-shuffles, combines, synthesises already existing facts, ideas, faculties, skills. The more familiar the parts, the more striking the new whole."

Koestler therefore presents creativity as a process of pattern formation, using components that already exist, but have not been combined in that particular way before. Accordingly, in so far as novelty is present, it is at the level of the pattern, rather than the components from which that pattern is formed. Beethoven, for example, did not invent the notes or the musical instruments that played them, but he did create some truly magical patterns. And each of Beethoven's successive musical patterns was necessarily different from all previous patterns.

That suggests that a process for the discovery of ideas—creativity—might be to take an existing construct, and then ask 'How might [this] be different?' of any specific feature (that's curiosity), the consequence of that question being to trigger exploration. And if, during that exploration, there is a willingness to unlearn, then perhaps ideas will emerge.

To test this proposal, here is a thought experiment that applies this process to the discovery of quaternions.

The starting point is Hamilton's wish "to possess a Theory of Triplets, analogous to my published Theory of Couplets". The "Theory of Couplets"—complex numbers—is therefore Hamilton's starting point, of which these are the key features with which Hamilton would have been familiar:

(1) A complex number $a + \mathbf{i}b$ has two parts: a real part, a, and an imaginary part, b.

(2) $|z_1 z_2| = |z_1||z_2|$ for any complex numbers z_1 and z_2 (the law of moduli).

(3) $\mathbf{i}^2 = -1$.

(4) Multiplication of complex numbers is associative.

(5) Multiplication of complex numbers is commutative.

How might each of these be different, and where might that lead?

The simplest possible, and most obvious, difference is to add a third term, changing $a + \mathbf{i}b$ to $a + \mathbf{i}b + \mathbf{j}c$, with $\mathbf{j}^2 = -1$ just as $\mathbf{i}^2 = -1$. But when Hamilton tested the other properties, he identified the problem with the product $\mathbf{ij}$—which, in a wonderful example of unlearning, he resolved by throwing commutative multiplication away and hypothesizing that $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. This in turn triggered another 'How might [this] be different?' question, where, in the relevant context, [this] was 'my assumption that there are three terms'. The most obvious answer is 'suppose there are four terms $a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d$', and when Hamilton tested that, it all worked.

Inspired by Hamilton's letter, John T. Graves discovered the 8-dimensional octonions in December 1843, as did Cayley, independently, in 1845. Like quaternions, octonions are non-commutative under multiplication, but unlike real numbers, complex numbers and quaternions, octonions are also non-associative under multiplication—this being a further example of asking 'How might [this] be different?', this time of the feature of the precedent quaternions that multiplication is associative.

It is of course most unlikely that Hamilton, Graves and Cayley actually discovered quaternions and octonions in this way. But they might have done so. Asking 'How might [this] be different?' of all the features of something that exists now is a hugely powerful way of discovering ideas, for the question gets to the heart of what creativity is: something different from, and hopefully better than, something that exists now.

Hamilton's discovery of the quaternions is just one of many examples of creativity in mathematics that fit the pattern of combining pre-existing components in different ways. Others include Andrew Wiles's proof of Fermat's Last Theorem, Henri Poincaré's discovery of automorphic forms, and Olga Taussky's work on determinantal conditions for matrix nonsingularity [9, Chap. 3]. Furthermore, seventy years of research on iterative refinement for the numerical solution of linear systems can be interpreted as arising from changing different features of the basic algorithm programmed by Wilkinson in 1948 for the Pilot ACE computer at the National Physical Laboratory [10].

In considering 'How might [this] be different?' in any context, we can think about a variety of attributes, such as:

- Size: can some quantity be bigger or smaller? In the quaternion example, we need 4 rather than 3 components. Often, thinking much bigger or much smaller (a quantity tending to infinity or to zero) will trigger useful ideas.
- Sequence: in a process involving a sequence of steps, do those steps have to be done in a certain order or can they be reordered or some steps even be merged?
- Established practice: if a property or condition is conventionally assumed, can it nevertheless be modified or dropped?

Creativity Workshops

In 2010, the Engineering and Physical Sciences Research Council (EPSRC) started the Creativity@home initiative[a] and encouraged PIs of large grants to request funds to run workshops on creative problem solving for their teams. The authors met at such a workshop that year and have since collaborated on several creativity workshops, Our creativity workshops consist of up to 20 people working in groups of up to 8, usually meeting for two days at an off-site location. They invariably generate a large number of ideas for later evaluation. These workshops are an excellent way to train people in creativity and help build a team environment in which creativity flourishes [9]. We have run some creativity workshops virtually in the last couple of years, and while they were successful we think that face to face events work better for all concerned. Our experience is consistent with recent research that reports that "videoconferencing hampers idea generation because it focuses communicators on a screen, which prompts a narrower cognitive focus" [1].

[a] bit.ly/ukri-creativity-at-home

Asking 'How might [this] be different?' is not, of course, the only way to generate ideas. In mathematics, three sources predominate. George Pólya's classic, *How to Solve It*, originally published in 1945, identifies no fewer than 70 problem-solving heuristics [14]. *The Mathematician's Mind* by Jacques Hadamard, also first published in 1945, is perhaps more philosophical than operational [5]. And Henri Poincaré's masterful 1908 lecture, *Mathematical Creation*, provides a highly personal insight into how he discovered Fuchsian functions [13]. In a wider context, authors such as Edward de Bono [3], Alex Osborn (to whom 'brainstorming' is attributed) [12], and Arthur van Gundy [16] are among the most notable. Asking 'How might [this] be different?', however, is easy, simple, and pragmatic.

The process for generating ideas that we have outlined is the basis of a six-step process that one of us has developed over the last twenty years [9], [15]. It can be applied to any focus of attention for which one wants to generate ideas and is well suited to tackling mathematical problems.

While the process can be carried out by individuals, it is at its most powerful when used in small groups. Creativity is much richer when people speak to each other, ask each other questions, and share knowledge. A group can spot more features of the topic under consideration and can produce a wider selection of ways in which those features could be different than any individual is likely to do.

Mathematicians are creative people, as the history of our subject shows. The process of listing all the properties of the problem at hand and asking 'How might [this] be different?' for each one, whether carried out by individuals or groups, can generate ideas that might otherwise be missed—and it's fun to apply it!

FURTHER READING

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For the last 20 years, Dennis Sherwood has been running The Silver Bullet Machine Manufacturing Company Limited, specializing in creativity and innovation. Previously, he was a consulting partner for Deloitte Haskins + Sells and Coopers & Lybrand, an Executive Director at Goldman Sachs, and Managing Director of the UK operations of SRI Consulting. He is the author of many articles and of 15 books.