Optimizing and Factorizing the Wilson Matrix

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Abstract. The Wilson matrix, W, is a 4×4 unimodular symmetric positive definite matrix of integers that has been used as a test matrix since the 1940's, owing to its mild ill-conditioning. We ask how close W is to being the most ill-conditioned matrix in its class, with or without the requirement of positive definiteness. By exploiting the matrix adjugate and applying various matrix norm bounds from the literature we derive bounds on the condition numbers for the two cases and we compare them with the optimal condition numbers found by exhaustive search. We also investigate the existence of factorizations $W = Z^T Z$ with Z having integer or rational entries. Drawing on recent research that links the existence of these factorizations to number-theoretic considerations of quadratic forms, we show that W has an integer factor Z and two rational factors, up to signed permutations. This little 4×4 matrix continues to be a useful example on which to apply existing matrix theory as well as being capable of raising challenging questions that lead to new results.

1. INTRODUCTION. In the early days of digital computing there was much interest in constructing matrices that could be used to test methods for solving linear systems and computing eigenvalues. Such matrices should have known inverse or eigenvalues, preferably of a simple form. A famous example is the Hilbert matrix, with (i, j) element $1/(i + j - 1)$, which was the subject of much investigation and about which a great deal is known [3], [14, Section 28.1]. This and other test matrices have been collected in books [10], [31, Appendix C] and made available in software, such as in MATLAB [12, Section 5.1] and Julia [34].

While the Hilbert matrix is defined for any dimension, some matrices of a specific small dimension have been proposed. Among these is the Wilson matrix

$$W = \begin{bmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{bmatrix},$$

which was a favorite of John Todd [28–30] and has been used by various authors, for example in [1, 6, 8, 9, 13, 17]. The earliest appearance we know of the Wilson matrix is in a 1946 paper by Morris [23], who investigates a linear system containing the matrix "devised by Mr. T. S. Wilson." A 1948 report by T. S. Wilson [32] acknowledges "Capt. J. Morris of the Royal Aircraft Establishment," so it appears that this is the same T. S. Wilson—an employee of D. Napier and Son, a British engineering company developing aircraft engines at the time (see also [33]).

Matrices with integer entries are of particular interest as test matrices because they are stored exactly in floating-point arithmetic, provided that the entries are not too large. By contrast, the Hilbert matrix is not stored exactly, and this can lead to difficulties in interpreting the results of computational experiments, as explained by Moler [20].

The Wilson matrix is symmetric positive definite and has determinant $\det(W) = 1$ (so it is unimodular[1]) and inverse

$$W^{-1} = \begin{bmatrix} 68 & -41 & -17 & 10 \\ -41 & 25 & 10 & -6 \\ -17 & 10 & 5 & -3 \\ 10 & -6 & -3 & 2 \end{bmatrix}.$$

The Wilson matrix is mildly ill-conditioned, with 2-norm condition number $\kappa_2(W) = \|W\|_2\|W^{-1}\|_2 \approx 2.98409 \times 10^3$, where $\|A\|_2 = \max_{x \neq 0} \|Ax\|_2/\|x\|_2$ and $\|x\|_2 = (x^T x)^{1/2}$. Recall that $\kappa_2(A) \geq 1$ and that $\kappa_2(A)$ is a measure of the sensitivity of a linear system $Ax = b$ to perturbations in A and b. Matrices with a large condition number are of interest for test purposes as they can pose various difficulties for methods for solving linear systems and other problems.

We do not know how Wilson, working in the pre-digital computer era, constructed his matrix, and in particular to what extent he maximized the condition number subject to the matrix entries being small integers. Moler [**21**] asked how ill-conditioned W is relative to matrices in the set

$$\mathcal{S} = \{ A \in \mathbb{R}^{4 \times 4} : A \text{ is nonsingular and symmetric with integer entries}$$

$$\text{between 1 and 10} \}. \tag{1}$$

He carried out an experiment in which he generated one million random matrices from $\mathcal{S}$. About 0.21 percent of the matrices had a larger condition number than that of W. The matrix with the largest condition number was

$$A_1 = \begin{bmatrix} 1 & 3 & 10 & 10 \\ 3 & 4 & 8 & 9 \\ 10 & 8 & 3 & 9 \\ 10 & 9 & 9 & 3 \end{bmatrix}, \tag{2}$$

which is not positive definite and has $\kappa_2(A_1) \approx 4.80867 \times 10^4$, determinant 1, and inverse

$$A_1^{-1} = \begin{bmatrix} 573 & -804 & 159 & 25 \\ -804 & 1128 & -223 & -35 \\ 159 & -223 & 44 & 7 \\ 25 & -35 & 7 & 1 \end{bmatrix}.$$

We will investigate the questions of what are the most ill-conditioned matrices in $\mathcal{S}$ and what are the most ill-conditioned matrices in $\mathcal{P}$, the subset of $\mathcal{S}$ comprising the positive definite matrices

$$\mathcal{P} = \{ A \in \mathbb{R}^{4 \times 4} : A \text{ is symmetric positive definite with integer entries between}$$

$$\text{1 and 10} \}. \tag{3}$$

We begin, in Section 2, by obtaining upper bounds on the condition numbers for these two cases. In Section 3 we determine the maximal condition numbers experimentally, by an exhaustive search.

Wilson may have constructed W as the product $Z^T Z$, where Z is a simpler integral matrix (one with integer entries). In Section 4 we identify a block triangular integral

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[1]A unimodular matrix is one with integer entries and determinant ± 1.

factor Z. By exploiting recent research that links the existence of these factorizations to number-theoretic considerations of quadratic forms, we show that W has not only an integral factor Z but also two rational factors, up to signed permutations. Using this new theory we also identify an integer factor and two rational factors of the most ill-conditioned matrix in $\mathcal{P}$.

The Wilson matrix may only be 4×4, but it raises some interesting challenges. This should not be surprising because Taussky noted in 1961 that "matrices with integral elements have been studied for a very long time and an enormous number of problems arise, both theoretical and practical." [26] Our study is an example of work on what have recently been termed "Bohemian matrices," defined as families of matrices whose entries are drawn from a finite discrete set, typically made up of small integers [5]. For other recent results on this topic, see [2, 7].

2. CONDITION NUMBER BOUNDS.

We wish to obtain upper bounds on $\kappa_2(A)$ for $A \in \mathcal{S}$, where $\mathcal{S}$ is defined in (1). We therefore need to bound $\|A\|_2$ and $\|A^{-1}\|_2$. First we consider $\|A\|_2$. We will use the inequality $\|A\|_2 \leq \|A\|_F$, where the Frobenius norm is given by $\|A\|_F = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}$. Equality in this inequality holds only for the zero matrix and rank-1 matrices, so since $A \in \mathcal{S}$ is nonsingular we have strict inequality. Nonsingularity also implies that matrices in $\mathcal{S}$ must have at least three entries not equal to 10, and they include, for example,

$$
\begin{bmatrix} 10 & 10 & 10 & 10 \\ 10 & 9 & 10 & 10 \\ 10 & 10 & 9 & 10 \\ 10 & 10 & 10 & 9 \end{bmatrix}, \quad
\begin{bmatrix} 10 & 10 & 10 & 10 \\ 10 & 9 & 10 & 10 \\ 10 & 10 & 10 & 0 \\ 10 & 10 & 9 & 10 \end{bmatrix}.
$$

(Both of these matrices have 2-norm condition number approximately 1.5×10^2, so they are quite well-conditioned.) Hence

$$
A \in \mathcal{S} \implies \|A\|_2 < \|A\|_F \leq (13 \times 100 + 3 \times 81)^{1/2} = \sqrt{1543}. \tag{4}
$$

Bound 1. We now derive a bound on $\|A^{-1}\|_2$ from first principles. The inverse of $A \in \mathbb{R}^{n \times n}$ is given by $A^{-1} = \operatorname{adj}(A)/\det(A)$, where the adjugate matrix

$$
\operatorname{adj}(A) = \left((-1)^{i+j} \det(A_{ji})\right),
$$

with A_{ij} denoting the submatrix of A obtained by deleting row i and column j. Since $|\det(A)| \geq 1$ for $A \in \mathcal{S}$,

$$
|A^{-1}| \leq |\operatorname{adj}(A)|, \tag{5}
$$

where the absolute value and the inequality are taken componentwise. We note in passing that a nonsingular integral matrix $A \in \mathbb{R}^{n \times n}$ has an integral inverse if and only if $\det(A) = \pm 1$. Hence if A and A^{-1} have integer entries then (5) is an equality.

Since $A \in \mathcal{S}$ is nonsingular, every 3×3 submatrix of $A \in \mathcal{S}$ must contain at least one entry less than or equal to 9. Hence from (5) we have, for $A \in \mathcal{S}$,

$$
\left|\left(A^{-1}\right)_{ij}\right| \leq |\det(A_{ji})| < \left(10 \times 3^{1/2}\right)^2 \times \sqrt{281}, \tag{6}
$$

where we have used Hadamard's inequality [16, Corollary 7.8.3], which states that for $B \in \mathbb{R}^{n \times n}$, $|\det(B)| \leq \prod_{k=1}^{n} \|b_k\|_2$, where b_k is the kth column of B. The inequality is strict because Hadamard's inequality is attained only for matrices with mutually

orthogonal columns, and no A_{ij} has this property since $A \in \mathcal{S}$ has positive elements. Using

$$\|B\|_2 \le n \max_{i,j} |b_{ij}| \tag{7}$$

[**14**, Section 6.2] with (6) and applying (4) we have

$$\kappa_2(A) < \sqrt{1543} \times 4 \times (10 \times 3^{1/2})^2 \times \sqrt{281} \approx 7.90164 \times 10^5, \tag{8}$$

which is our first bound on the condition number of $A \in \mathcal{S}$.

Bound 2. Richter [**24**] and Mirsky [**19**] proved that for $B \in \mathbb{R}^{n \times n}$,

$$\|\operatorname{adj}(B)\|_F \le \frac{\|B\|_F^{n-1}}{n^{(n-2)/2}},$$

which implies

$$\kappa_F(B) \le \frac{\|B\|_F^n}{n^{(n-2)/2}|\det(B)|}.$$

For $A \in \mathcal{S}$, using (4), we obtain

$$\kappa_2(A) < \kappa_F(A) \le \frac{1543^2}{4} \approx 5.95212 \times 10^5, \tag{9}$$

which is a useful improvement on (8). The proof in [**19**] makes use of an inequality for elementary symmetric functions applied to the singular values of A.

Bound 3. We can obtain another bound using a result of Guggenheimer, Edelman, and Johnson [**11**], which is obtained by applying the arithmetic-geometric mean inequality to a function of the singular values. The result states that for nonsingular $B \in \mathbb{R}^{n \times n}$,

$$\kappa_2(B) < \frac{2}{|\det(B)|} \left(\frac{\|B\|_F}{n^{1/2}} \right)^n. \tag{10}$$

Using (4) gives, for $A \in \mathcal{S}$,

$$\kappa_2(A) < 2 \left(\frac{\sqrt{1543}}{2} \right)^4 \approx 2.97606 \times 10^5, \tag{11}$$

which improves on (9) by a factor 2.

Bound 4. We can also use the bound for nonsingular $B \in \mathbb{R}^{n \times n}$ of Merikoski, Urpala, Virtanen, Tam, and Uhlig [**18**, Theorem 1],

$$\kappa_2(B) \le \left(\frac{1+x}{1-x} \right)^{1/2}, \quad x = \sqrt{1 - (n/\|B\|_F^2)^n |\det(B)|^2}. \tag{12}$$

Applying this bound to $A \in \mathcal{S}$, using the fact that $(1+x)/(1-x)$ is monotonically increasing for $x \in (0, 1)$, gives

$$\kappa_2(A) \le 2.97606\ldots \times 10^5. \tag{13}$$

Computing at 100-digit precision shows that the bound (13) is approximately 3.36×10^{-6} smaller than (11), so this is our best bound! It is shown in [**18**] that the bound (12) is the smallest bound that can be obtained based on $\|B\|_F$, $\det(B)$, and n only, so to obtain a better bound on $\kappa_2(A)$ we would need to exploit symmetry or the integer nature of the entries.

Bound for the positive definite case. Let us now add a definiteness requirement and maximize over the set $\mathcal{P}$ in (3), which comprises the positive definite matrices in $\mathcal{S}$. Can we obtain a smaller condition number bound over $\mathcal{P}$ than over $\mathcal{S}$?

We can modify the derivation of the first bound. Since A is positive definite, $\mathrm{adj}(A) = \det(A)A^{-1}$ is positive definite and so its largest element lies on the diagonal. Therefore $\|\mathrm{adj}(A)\|_2 \leq 4\max_i |(\mathrm{adj}(A))_{ii}|$ by (7), and $|(\mathrm{adj}(A))_{ii}| = |\det(A_{ii})| \leq 10^3$ by Hadamard's inequality ($\det(C) \leq c_{11}\cdots c_{nn}$ for a symmetric positive definite $C \in \mathbb{R}^{n\times n}$). Hence, using (4),

$$\kappa_2(A) \leq \|A\|_2 \|\mathrm{adj}(A)\|_2 \leq \sqrt{1543} \times 4 \times 10^3 \approx 1.57124 \times 10^5, \qquad (14)$$

which improves on our best bound (13) for the general symmetric case.

We can obtain another upper bound for $\kappa_2(A)$ from [**18**, Theorem 2] (a bound for matrices with real, positive eigenvalues) which, when specialized to symmetric positive definite matrices $C \in \mathbb{R}^{n\times n}$, gives

$$\kappa_2(C) \leq \frac{1+x}{1-x}, \quad x = \sqrt{1 - (n/\mathrm{trace}(C))^n \det(C)}.$$

For $A \in \mathcal{P}$ we obtain, since $\det(A) \geq 1$,

$$\kappa_2(A) \leq \frac{1+x}{1-x}, \quad x = \sqrt{1 - (1/10)^4},$$

which evaluates to

$$\kappa_2(A) \leq 3.99980 \times 10^4. \qquad (15)$$

The bound (15) is a significant improvement on (14).

Table 1 summarizes all the bounds we have obtained.

3. EXPERIMENTS. It is possible to maximize $\kappa_2(A)$ over $\mathcal{S}$ or $\mathcal{P}$ by exhaustively searching the whole set with a computer program. These are large sets: $\mathcal{S}$ has 10^{10} elements. In order to minimize the computation time of several hours one needs to optimize the code. This can be done by looping over all members A of $\mathcal{S}$ or $\mathcal{P}$ and

Table 1. Upper bounds, to six significant figures, for $\kappa_2(A)$, for A in the set specified in the first column, which is $\mathcal{S}$ in (1) or $\mathcal{P}$ in (3).

Set	Bound	Value
$\mathcal{S}$	(8)	7.90164×10^5
$\mathcal{S}$	(9)	5.95212×10^5
$\mathcal{S}$	(11)	2.97606×10^5
$\mathcal{S}$	(13)	2.97606×10^5
$\mathcal{P}$	(14)	1.57124×10^5
$\mathcal{P}$	(15)	3.99980×10^4

- evaluating det(A) from an explicit expression (exactly computed for such matrices) and discarding A if the matrix is singular;
- computing the eigenvalues λ_i of A and obtaining the condition number as $\kappa_2(A) = \max_i |\lambda_i| / \min_i |\lambda_i|$ (since A is symmetric); and
- for $\mathcal{P}$, checking whether A is positive definite by checking whether the smallest eigenvalue is positive.

Most of the time is spent computing eigenvalues. Our computation was done in MATLAB R2020a on a PC with an Intel Core i7-6800K processor and took just under six hours to find both maximizers. The computation can be sped up by using explicit formulas for the eigenvalues instead of calling the MATLAB eigensolver eig (which is not optimized for small matrices), but the explicit formulas are difficult to implement in a numerically reliable way. The code we used is available at https://github.com/higham/wilson-opt.

The maximum over $\mathcal{S}$ is attained for the matrix

$$A_2 = \begin{bmatrix} 2 & 7 & 10 & 10 \\ 7 & 10 & 10 & 9 \\ 10 & 10 & 10 & 1 \\ 10 & 9 & 1 & 9 \end{bmatrix},$$

which has $\kappa_2(A) \approx 7.6119 \times 10^4$, determinant -1, and inverse

$$A_2^{-1} = \begin{bmatrix} 640 & -987 & 323 & 240 \\ -987 & 1522 & -498 & -370 \\ 323 & -498 & 163 & 121 \\ 240 & -370 & 121 & 90 \end{bmatrix}.$$

Symmetric permutations of A_2 are of course also maximizers. The condition number $\kappa_2(A_2)$ is a factor 3.9 smaller than the upper bound (13) and 25.5 times larger than $\kappa_2(W)$. We note that A_2 is clearly not positive definite because the 2×2 principal submatrices involving row and column 1 all have negative determinant. (In fact, the smallest eigenvalue of A_2 is -0.11.)

The maximum over $\mathcal{P}$ is attained for

$$A_3 = \begin{bmatrix} 9 & 1 & 1 & 5 \\ 1 & 10 & 1 & 9 \\ 1 & 1 & 10 & 1 \\ 5 & 9 & 1 & 10 \end{bmatrix}, \tag{16}$$

for which $\kappa_2(A_3) \approx 3.5529 \times 10^4$, $\det(A_3) = 1$, and

$$A_3^{-1} = \begin{bmatrix} 188 & 347 & -13 & -405 \\ 347 & 641 & -24 & -748 \\ -13 & -24 & 1 & 28 \\ -405 & -748 & 28 & 873 \end{bmatrix}.$$

We see that $\kappa_2(A_3)$ is a factor 1.126 smaller than the upper bound 3.9998×10^4 in (15) and 11.9 times larger than $\kappa_2(W)$.

We have also run experiments using the genetic optimization code ga from the MATLAB Global Optimization Toolbox to attempt to maximize $\kappa_2(A)$ subject to $A \in \mathcal{S}$ and $A \in \mathcal{P}$. This code requires only function values and not derivatives. With repeated runs, these experiments produced A_2 and A_3 after just a few minutes of computation time, but they provide no indication of optimality.

Table 2. Matrices and their condition numbers, to six significant figures. The matrix A_1 in (2) was found by Moler [21] by taking a million random samples from $\mathcal{S}$.

Matrix	Comment	$\kappa_2(A)$
W	Wilson matrix	2.98409×10^3
A_1	By random sampling	4.80867×10^4
A_2	Optimal matrices in $\mathcal{S}$	7.61190×10^4
A_3	Optimal matrices in $\mathcal{P}$	3.55286×10^4

Table 2 summarizes what we have found.

4. FACTORIZATIONS WITH INTEGER ENTRIES.

Every symmetric positive definite matrix A has a Cholesky factorization $A = R^T R$, where R is upper triangular with positive diagonal elements. The factorization is used in solving linear systems $Ax = b$. The Cholesky factor of the Wilson matrix does not have integer entries. Suppose we drop the requirement of triangularity and ask whether the Wilson matrix has a factorization $W = Z^T Z$ with a 4×4 matrix Z of integers. It is known that every symmetric positive definite $n \times n$ matrix A of integers with determinant 1 has a factorization $A = Z^T Z$ with Z an $n \times n$ matrix of integers as long as $n \leq 7$, but examples are known for $n = 8$ for which the factorization does not hold. This result is mentioned by Taussky [25, p. 336], [27, pp. 812–813] and goes back to Hermite, Minkowski, and Mordell [22].

By using the ga function to minimize $\|W - Z^T Z\|_F$ over integer matrices Z we found the integer factor

$$Z_0 = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{17}$$

for the Wilson matrix. This factor Z_0 is rather special: it has nonnegative elements and is block upper triangular, so it can be thought of as a block Cholesky factor.

The number-theoretic properties relating to the factorization of symmetric positive definite $n \times n$ integer matrices A with determinant D_n have classical connections to the theory of positive definite quadratic forms in n variables, as discussed by the authors mentioned above. In particular, Mordell considered two such $n \times n$ matrices A, B to be in the same class if there exists a unimodular integral matrix Z such that $A = Z^T B Z$, where the number of such classes is denoted by h_n.

In the case that $B = I_n$, the identity matrix, so that $D_n = 1$, we have that A is in the same class as I_n, and hence A factors as $A = Z^T Z$, with Z an integer matrix. The corresponding quadratic form associated with the matrix A can therefore be written as $(Zx)^T(Zx)$, a sum of squares of n linear factors. For the Wilson matrix this gives the quadratic form, for $x \in \mathbb{R}^4$,

$$q(x) = (2x_1 + 3x_2 + 2x_3 + 2x_4)^2 + (x_1 + x_2 + 2x_3 + x_4)^2$$
$$+ (x_3 + 2x_4)^2 + (x_3 + x_4)^2,$$

a sum of four squares. The coordinate transformation $y = Z_0 x$ is unimodular because $\det(Z_0)^2 = \det(W) = 1$, and it diagonalizes the quadratic form such that $q(y) = y_1^2 + y_2^2 + y_3^2 + y_4^2$. By Lagrange's four-square theorem, it follows that the quadratic form generated by the Wilson matrix is *universal* [4], i.e., it generates the positive integers,

so that for a given positive integer n there exists a vector y with nonnegative integer entries with $q(y) = n$.

In what follows, a key concept is the idea of the *weight* w_M of an $n \times m$ matrix M, defined to be the sum of its entries divided by the number of entries nm. If the matrix under consideration is square and has integer entries, then the matrix weight w_M is an integer divided by n^2, so that $w_M \in \frac{1}{n^2}\mathbb{Z}$. For the Wilson matrix the weight $w_W = \frac{119}{16}$.

Another important concept is the idea of the matrix M having a unique decomposition $M = M_V + M_S + w_M \mathcal{E}_n$ over the type V and type S matrix symmetry spaces [15], where $\mathcal{E}_n$ is the $n \times n$ matrix with every entry 1. We say that $M_V = (m_{ij})$ has the (type V) *vertex cross-sum property* if for $i \neq i'$ and $j \neq j'$, we have $m_{ij} + m_{i'j'} = m_{ij'} + m_{i'j}$ and that the sum of all the elements in M_V is zero. The matrix M_S has row and column sums all equal to zero, so that $M_S + w_M \mathcal{E}_n$ has constant row and column sums $n w_M$. Such a matrix is traditionally known as a *semi-magic square*.

In essence this decomposition splits the matrix M into a semi-magic square, $M_S + w_M \mathcal{E}_n$, with weight w_M, and the converse type V symmetry type matrix M_V described above. The direct sum of these two symmetry spaces $S \oplus V$ forms what is traditionally known as a $\mathbb{Z}_2$-graded algebra (also known as a superalgebra) which has a decomposition into an "even" subalgebra and an "odd" complementary part that is a bimodule over the "even" subalgebra and squares into it. Here the type V symmetry space is the "odd" part of the superalgebra, and the type S symmetry space the "even" part. The product of two type V matrices or two type S matrices is type S, whereas the product of a type V and a type S matrix is of type V. The type S and type V symmetry spaces are orthogonal with respect to the Frobenius norm.

Additionally, as established by the authors in conjunction with Schmidt [15], if M is symmetric, then in this decomposition the odd part M_V takes the form

$$M_V = a 1_n^T + 1_n a^T,$$

for some vector $a \in \mathbb{R}^n$, with 1_n the n-dimensional vector with every entry 1. The vector a can be obtained explicitly from the formula

$$a_i = \frac{1}{n} \sum_{j=1}^{n} m_{ij} - w_M, \quad i \in \{1, \ldots, n\},$$

so that a is perpendicular to 1_n. In particular, if M is an integer matrix, then the vector $n^2 a$ has integer entries.

As established in [15], a necessary (but not sufficient) condition for an integer matrix M to be factored as $M = Z^T Z$ is that the weights in the $S \oplus V$ decompositions of the matrix and the matrix factor equate. Setting $Z = Z_V + Z_S + w_Z \mathcal{E}_n$ with $Z_V = a 1_n^T + 1_n a^T$, we obtain the quadratic equation [15, Theorem 3.1]

$$w_M = |a|^2 + n (w_Z)^2. \tag{18}$$

For the factor Z_0 of the Wilson matrix we find that the $S \oplus V$ decomposition $Z_0 = Z_V + Z_S + w_{Z_0} \mathcal{E}_4$ is given by

$$Z_0 = Z_V + Z_S + w_{Z_0} \mathcal{E}_4$$

$$= \frac{1}{8} \begin{bmatrix} 5 & 7 & 11 & 11 \\ -3 & -1 & 3 & 3 \\ -7 & -5 & -1 & -1 \\ -9 & -7 & -3 & -3 \end{bmatrix} + \frac{1}{16} \begin{bmatrix} 3 & 15 & -9 & -9 \\ 3 & -1 & 7 & -9 \\ -5 & -9 & -1 & 15 \\ -1 & -5 & 3 & 3 \end{bmatrix} + \frac{19}{16} \mathcal{E}_4,$$

so that Z_0 has weight $w_{Z_0} = \frac{19}{16}$ and, as expected, the entries of Z_V and Z_S are in $\frac{1}{16}\mathbb{Z}$.

For the Wilson matrix, (18) gives the quadratic equation

$$\frac{119}{16} = 2(a_1^2 + a_2^2 + a_3^2 + a_1a_2 + a_1a_3 + a_2a_3) + 4(w_Z)^2,$$

where we have set $n = 4$, $a_4 = -a_1 - a_2 - a_3$ in (18). Multiplying through by 4^4, setting $x_i = 16a_i$, $w = 16w_Z$, and then dividing through by 2, we find that a necessary condition for the Wilson matrix to factor as $W = Z^T Z$, is for integer solutions to exist to the quadratic equation

$$2w^2 + x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 = 952. \qquad (19)$$

Solving for w, x_1, x_2, x_3 in *Mathematica* 11.0 on a PC with a Intel Core i7 6500CPU, for nonnegative weights w, we found 1728 solutions in just under 6 seconds. Substituting each integer solution to (18) into the overall matrix factorization problem (discussed in more detail in [**15**]) and solving took considerably longer at 34 minutes, yielding 576 matrices Z with $W = Z^T Z$. Hence exactly one third of the solutions to (19) correspond to matrix factorization solutions.

For any two matrix factorizations Z_i, Z_j, with $W = Z_i^T Z_i = Z_j^T Z_j$, there exists a rational orthogonal matrix U with $Z_i = UZ_j$. However, if we restrict U to being a signed permutation matrix, then the factorization matrices can be grouped into three classes, represented by the matrix Z_0 and the two rational matrices Z_1 and Z_2, where

$$Z_0 = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 1 \\ \frac{3}{2} & 2 & 3 & 3 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 2 & 1 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \frac{3}{2} & 2 & 2 & 2 \\ \frac{3}{2} & 2 & 2 & 1 \\ \frac{1}{2} & 1 & 1 & 2 \\ -\frac{1}{2} & -1 & 1 & 1 \end{bmatrix}.$$

These three factorizations $W = Z_0^T Z_0 = Z_1^T Z_1 = Z_2^T Z_2$, respectively correspond to the (w, x_1, x_2, x_3) solutions to (19),

$$(19, 17, 1, -7), \qquad (18, -8, 20, -12), \qquad (19, 11, 7, -1), \qquad (20)$$

highlighting that it is possible for different matrix factorizations to have different weights. In total we found that there are 11 different positive weights for the factorization matrices. However, up to multiplication by signed permutation matrices, Z_0, Z_1 and Z_2, are the only rational matrix factors of W with entries in $\frac{1}{16}\mathbb{Z}$, highlighting the power of the analysis that this approach can offer.

In summary, we have shown how equating the weight of the Wilson matrix with the weights arising from the assumed factorization and using number-theoretic considerations leads to rational factorizations of the Wilson matrix.

We applied the same technique to the matrix A_3 in (16). Again, we found one integral factor and two rational factors (up to signed permutations):

$$Z_0 = \begin{bmatrix} -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 3 \\ 2 & 0 & -2 & 1 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} -\frac{3}{2} & 2 & 1 & 1 \\ -\frac{1}{2} & -1 & -2 & -1 \\ -\frac{1}{2} & -2 & 2 & -2 \\ \frac{5}{2} & 1 & 1 & 2 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} \frac{1}{2} & 1 & 3 & 1 \\ -\frac{3}{2} & 2 & 0 & 1 \\ \frac{5}{2} & 1 & 0 & 2 \\ \frac{1}{2} & 2 & -1 & 2 \end{bmatrix}.$$

No permutation of the integral factor is block triangular, however.

For larger matrices, this approach will usually become more complicated, due to the existence of a larger number of solutions to the quadratic form, arising from the matrix weights in the factorization.

5. CONCLUSIONS. Wilson's matrix W is a symmetric positive definite integer matrix with an integer inverse. It is mildly ill-conditioned, with $\kappa_2(W) \approx 2.9841 \times 10^3$, but is some way from being the most ill-conditioned matrix of its type: the extreme case has a condition number about 25 times larger, or about 12 times larger if positive definiteness is required.

The best upper bounds that we could derive for the condition number of matrices in $\mathcal{S}$ in (1) are factors 3.9 and 1.13 (with a positive definiteness constraint) larger than the worst-case condition numbers, and so are reasonably sharp.

The integer matrix factorization $W = Z_0^T Z_0$, with the block upper triangular Z_0 in (17), demonstrates that the quadratic form $x^T W x$ arising from the Wilson matrix is universal, representing all positive integers. When this matrix factorization in generality is considered in terms of the quadratic forms arising from the matrix weights, we find that, ignoring permutations and sign changes, there are fundamentally only three rational factorization matrices with entries in $\frac{1}{16}\mathbb{Z}$.

Despite its modest dimensions, the Wilson matrix has generated a lot of interest over the more than seventy years since it was proposed, and we have been able to shed more light on its properties. Regarding our question in Section 1 about how Wilson constructed his matrix, we think it most likely that he formed it via the block triangular matrix Z_0 in (17).

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An "Esoteric" Proof of Gelin-Cesàro Identity

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. A proof by induction shows that they satisfy Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, see D. E. Knuth [1, p. 81]. A more esoteric way to prove the same formula starts with a simple inductive proof of the matrix identity

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$

We can then take the determinant of both sides of this equation.

The relation

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1$$

is known as the Gelin-Cesàro identity (proposed by Gelin as **Question 570** in the *Nouvelle Correspondance Mathématique* in 1880, p. 384, and solved by Cesàro in the same journal later in 1880, pp. 423–424). Here we present an "esoteric" proof of it. Indeed, using that $F_j^2 - F_{j-1}^2 = (F_j - F_{j-1})(F_j + F_{j-1}) = F_{j-2}F_{j+1}$, we can check the formula

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n-1}F_{n+2} & F_n^2 \\ F_n^2 & F_{n-2}F_{n+1} \end{pmatrix}.$$

We finish the proof of Gelin-Cesàro identity by taking determinants on both sides of the previous equation.

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