The Incidence Matrix of a Graph
Nullspaces, Rank, Pseudoinverse

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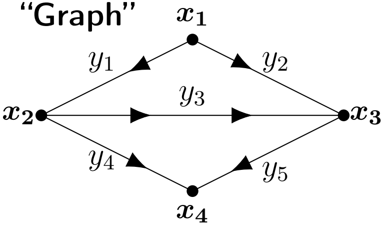


$$\begin{array}{cccccc} \textbf{nodes} & 1 & 2 & 3 & 4 & \textbf{edges} \end{array}$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{1} & 0 & 0 \\ \mathbf{-1} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{-1} & \mathbf{1} & 0 \\ 0 & \mathbf{-1} & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{-1} & \mathbf{1} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Graph has $\mathbf{5}$ edges, $\mathbf{4}$ nodes. $\boldsymbol{A}$ is its $\mathbf{5 \times 4}$ incidence matrix. Rank $= \mathbf{3}$

Columns $1, 2, 3$ are a basis for **C**(A) Column $4 = -$ Columns $1 + 2 + 3$

$\boldsymbol{y} = y_1$ to $y_5 =$ currents on edges $\boldsymbol{x} = x_1$ to $x_4 =$ voltages at nodes

Edges $\mathbf{1, 2, 3}$ form a **loop** in the graph	**Dependent rows $\mathbf{1, 2, 3}$**
Edges $\mathbf{1, 2, 4}$ form a **tree (no loops)**.	**Independent rows $\mathbf{1, 2, 4}$**

Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ at each node flow in $=$ flow out

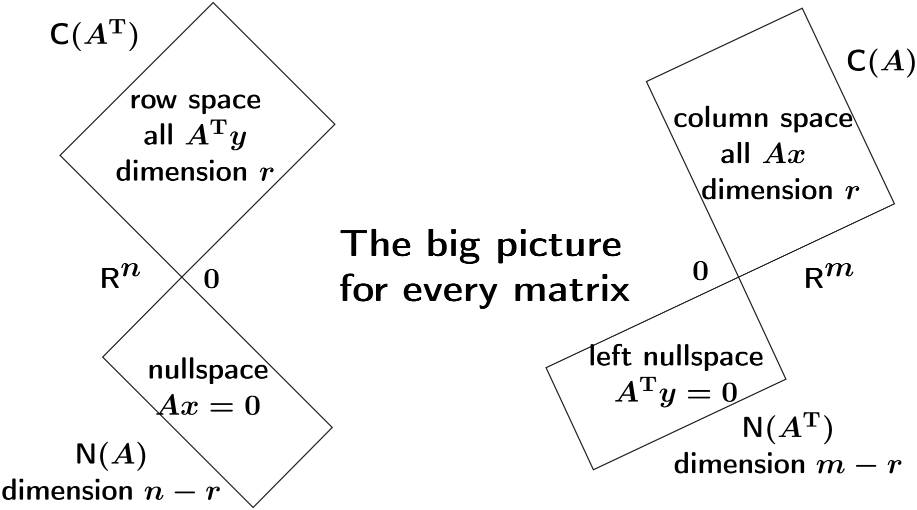
Elimination on A produces its echelon form R_0: **rank 3**

Reduced row echelon form of the incidence matrix

$$\boldsymbol{R_0} = \begin{bmatrix} \mathbf{1} & 0 & 0 & \mathbf{-1} \\ 0 & \mathbf{1} & 0 & \mathbf{-1} \\ 0 & 0 & \mathbf{1} & \mathbf{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column space basis: Columns $1, 2, 3$ of A
Row space basis: Rows $1, 2, 3$ of R_0
Nullspace basis: $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
Nullspace of A^{T}: 2 loops give basis
$(\mathbf{1}, \mathbf{-1}, \mathbf{1}, \mathbf{0}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{0}, \mathbf{-1}, \mathbf{1}, \mathbf{-1})$

Four Fundamental Subspaces $\mathsf{C}(A)$,$\mathsf{C}(A^{\mathrm{T}})$,$\mathsf{N}(A)$,$\mathsf{N}(A^{\mathrm{T}})$



The big picture for every matrix

Fundamental Theorem of Linear Algebra, Part 1

The column space and row space both have dimension r

The nullspaces have dimensions $n - r$ and $m - r$

m equations, n unknowns, rank r

$Ax = \mathbf{0}$ has $n - r$ independent solutions

There is always a nonzero solution x to $Ax = \mathbf{0}$ if $n > m$

Fundamental Theorem, Part 2: **Subspaces are orthogonal**
(Each row of A) $\cdot\, x = \mathbf{0}$

Fundamental Theorem, Part 3: **Perfect bases from singular vectors and the SVD**

Every A has a pseudoinverse A^+ $\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^+ = \begin{bmatrix} \frac{1}{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

A is invertible if and only if $m = n = r$ (rank). Then $\boldsymbol{A^+ = A^{-1}}$

A has a **left inverse** $\boldsymbol{A^+ = (A^T A)^{-1} A^T}$ when $\boldsymbol{r = n}$.
Then $\boldsymbol{A^+ A = I_n}$

A has a **right inverse** $\boldsymbol{A^+ = A^T (A A^T)^{-1}}$ when $\boldsymbol{r = m}$.
Then $\boldsymbol{A A^+ = I_m}$

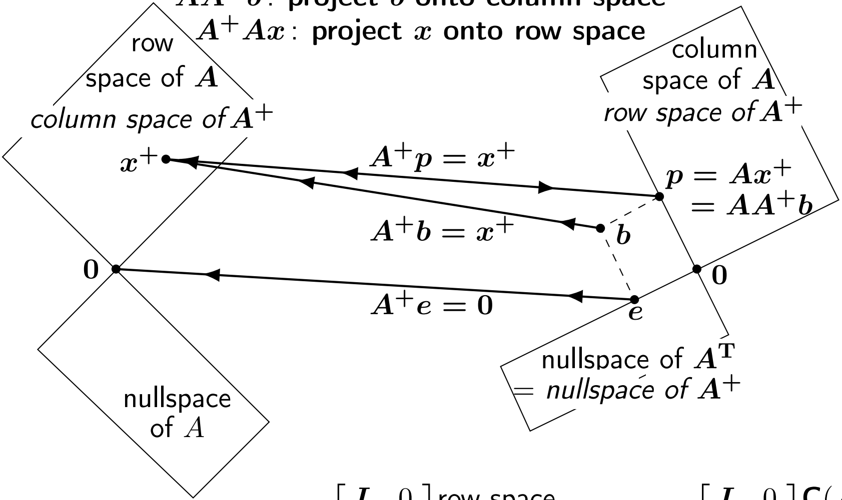
Every $\boldsymbol{A = CR}$ has the **pseudoinverse** $\boldsymbol{A^+ = R^+ C^+}$.
Just reverse the 4 subspaces !

A Row space to column space

A^+ Column space to row space

AA^+b: project b onto column space

A^+Ax: project x onto row space



Pseudoinverse A^+ $A^+A = \begin{bmatrix} I & 0 \\ 0 & \mathbf{0} \end{bmatrix}$ row space / nullspace $AA^+ = \begin{bmatrix} I & 0 \\ 0 & \mathbf{0} \end{bmatrix}$ $\mathbf{C}(A)$ / $\mathbf{N}(A^T)$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{1} & 0 & 0 \\ \mathbf{-1} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{-1} & \mathbf{1} & 0 \\ 0 & \mathbf{-1} & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{-1} & \mathbf{1} \end{bmatrix} \qquad \boldsymbol{A}^+ = \frac{1}{8}\begin{bmatrix} -3 & -3 & 0 & -1 & -1 \\ 2 & 0 & -2 & -2 & 0 \\ 0 & 2 & 2 & 0 & -2 \\ 1 & 1 & 0 & 3 & 3 \end{bmatrix}$$

Each row of A adds to zero — **Each column of A^+ adds to zero**

Row 2 of $A =$ Row $1 +$ Row 3 — Column 2 of $A^+ =$ Column $1 +$ Column 3

Row 4 of $A =$ Row $3 +$ Row 5 — Column 4 of $A^+ =$ Column $3 +$ Column 5

The 4 subspaces for $\boldsymbol{A}^{\mathbf{T}}$ are also the 4 subspaces for $\boldsymbol{A}^+$

How to compute the pseudoinverse A^+ ?

$\mathbf{1}$	Use the SVD $\boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathbf{T}}$	Then $\boldsymbol{A}^+ = \boldsymbol{V\Sigma}^+\boldsymbol{U}^{\mathbf{T}}$
$\mathbf{2}$	Use modified Gram-Schmidt $\boldsymbol{A} = \boldsymbol{QR}$	Then $\boldsymbol{A}^+ = \boldsymbol{R}^{-\mathbf{1}}\boldsymbol{Q}^{\mathbf{T}}$
$\mathbf{3}$	Use column-row factorization $\boldsymbol{A} = \boldsymbol{CR}$	Then $\boldsymbol{A}^+ = \boldsymbol{R}^+\boldsymbol{C}^+$

Question   Is A^+ rational when the entries of A are rational ?

Key fact: **The pseudoinverse of AB is not always B^+A^+** but

$(\boldsymbol{CR})^+ = \boldsymbol{R}^+\boldsymbol{C}^+ = \boldsymbol{R}^{\mathbf{T}}(\boldsymbol{RR}^{\mathbf{T}})^{-\mathbf{1}}(\boldsymbol{C}^{\mathbf{T}}\boldsymbol{C})^{-\mathbf{1}}\boldsymbol{C}^{\mathbf{T}}$ is true because

$\boldsymbol{C}$ has full column rank and $\boldsymbol{R}$ has full row rank in $\boldsymbol{A} = \boldsymbol{CR}$

Graph incidence matrix (and every matrix) $\boldsymbol{A} = \boldsymbol{CR}$

$$\boldsymbol{C} = \textbf{first } \boldsymbol{r} \textbf{ independent columns of } \boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\boldsymbol{R} = \boldsymbol{r}$ **nonzero rows of rref**(A)

$\boldsymbol{R_0} =$ **reduced row echelon form**

$$= \begin{bmatrix} \mathbf{1} & 0 & 0 & \mathbf{-1} \\ 0 & \mathbf{1} & 0 & \mathbf{-1} \\ 0 & 0 & \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

$\boldsymbol{A}^+ = \boldsymbol{C}^+\boldsymbol{R}^+ =$ rational computation of the pseudoinverse

Complete graph with $\mathbf{4}$ nodes and $\mathbf{6}$ edges
$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} =$ Graph Laplacian matrix

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{L} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & -1 & -1 & -1 \\ -1 & \mathbf{3} & -1 & -1 \\ -1 & -1 & \mathbf{3} & -1 \\ -1 & -1 & -1 & \mathbf{3} \end{bmatrix}$$

$L = A^{\mathrm{T}}A$ has eigenvalues $4, 4, 4, 0$. Its first 3 eigenvectors are orthogonal to its 4th eigenvector $(1, 1, 1, 1)$.
Then $\boldsymbol{P} = \boldsymbol{L}/\mathbf{4}$ has eigenvalues $1, 1, 1, 0$: a **projection matrix**.

For complete graphs, the pseudoinverse A^{+} is the same as $\dfrac{\mathbf{1}}{\boldsymbol{n}}\boldsymbol{A}^{\mathbf{T}}$

$$\boldsymbol{A^+} = \textbf{Pseudoinverse} = \frac{1}{4}\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{A}^{\mathbf{T}}$$

$\boldsymbol{\Sigma}^{\boldsymbol{+}} = \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{\Sigma}^{\mathbf{T}}$. Then $\boldsymbol{A}^+ = (U\Sigma V^{\mathrm{T}})^+ = V\boldsymbol{\Sigma}^{\boldsymbol{+}}U^{\mathrm{T}} = \frac{\mathbf{1}}{\mathbf{4}}V\boldsymbol{\Sigma}^{\mathbf{T}}U^{\mathrm{T}} = \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{A}^{\mathbf{T}}$

$$A^{\mathrm{T}}A = \frac{1}{16}\begin{bmatrix} \mathbf{3} & -1 & -1 & -1 \\ -1 & \mathbf{3} & -1 & -1 \\ -1 & -1 & \mathbf{3} & -1 \\ -1 & -1 & -1 & \mathbf{3} \end{bmatrix}$$

The pseudoinverse is $\dfrac{A^{\mathrm{T}}A}{16}$

$A^{\mathrm{T}}A$ has eigenvalues $\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{0}$. Therefore $(A^{\mathrm{T}}A)^+$ has eigenvalues $\frac{\mathbf{1}}{\mathbf{4}}, \frac{\mathbf{1}}{\mathbf{4}}, \frac{\mathbf{1}}{\mathbf{4}}, \mathbf{0}$.

The eigenvectors are the same so $\boldsymbol{(A^{\mathbf{T}}A)^{+}} = \frac{\mathbf{1}}{\mathbf{16}}\boldsymbol{(A^{\mathbf{T}}A)}$.

The Column-Row Factorization $A = CR$

A new start for linear algebra

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Linear Algebra for Everyone (2020)

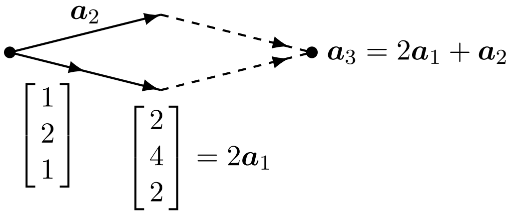
$$A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{2} & \mathbf{3} & \mathbf{7} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} \end{bmatrix} \qquad \begin{matrix} m = 3 \text{ rows} \\ n = 3 \text{ columns} \end{matrix}$$

Are the columns independent ? Go left to right

Column 1 OK Column 2 OK Column 3 ?

Column $3 = 2\,$(Column 1) $+1\,$(column 2) **Dependent**

Column 3 is in the plane of Columns 1 and 2



Matrix $C = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}$ of independent columns in $A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{2} & \mathbf{3} & \mathbf{7} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} \end{bmatrix}$

The matrix A has **column rank** $\boldsymbol{r = 2}$

The **column space** of A is a plane in $\mathbf{R}^3$

The column space contains all combinations of the columns

Column space of $A =$ Column space of C ((but $A \neq C$))

Express the steps by multiplications Ax and CR

Ax = **matrix times vector** = **combination of columns of** $\boldsymbol{A}$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2\,(\text{Column } 1) + 1\,(\text{Column } 2) - 1\,(\text{Column } 3)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{dot products of } \boldsymbol{x} \text{ with rows of } A)$$

$\boldsymbol{CR}$ = **Matrix times matrix** = $\boldsymbol{C}$ **times each column of** $\boldsymbol{R}$

Use dot products (low level) or take combinations of the columns of C

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \mathbf{2} \\ 0 & 1 & \mathbf{1} \end{bmatrix} \quad \text{is } \boldsymbol{A = CR}$$

Check C times each column of R

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\,(\text{Column } 1) + (\text{Column } 2) = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$$

$$2\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a}_3$$

How to find CR for every A? **Elimination !**

$\boldsymbol{A} = \boldsymbol{CR}$ is $(m$ by $n) = (m$ by $r)\,(r$ by $n)$

$\boldsymbol{R} = \left[\ \boldsymbol{I} \quad \boldsymbol{F}\ \right] \boldsymbol{P}$ and $\boldsymbol{A} = \boldsymbol{CR} = \left[\ \boldsymbol{C} \quad \boldsymbol{CF}\ \right] \boldsymbol{P}$

In reality we compute R before C !! The columns of I in R tell us the independent columns of A in C.

The permutation P puts those columns in the right places (if they are not the first r columns of A)

$\boldsymbol{R} =$ **reduced row echelon form rref**(A) **(zero rows removed)**

Here are the steps to establish $A = CR$

We know $EA = \textbf{rref}(A)$ and $A = E^{-1}\,\textbf{rref}(A)$: E is $m \times m$

Remove $m - r$ zero rows from $\textbf{rref}(A)$ and $m - r$ columns from E^{-1}

This leaves $A =$ (some matrix C) times (known form $\begin{bmatrix} I & F \end{bmatrix} P$)

C must hold independent columns $\quad$ CF holds dependent columns

C has r independent columns R has r independent rows

Rows of $A = CR$ are combinations of the rows of R

Row space of A = Row space of R ! (from $A = CR$)

If A has 2 independent columns in C then **A has 2 independent rows**

Column rank = Row rank = r GREAT THEOREM

Look at $A = CR$ both ways: Combine columns of C Combine rows of R

$\boldsymbol{r = 1}$ Rank one matrix $A = (1 \text{ column})\,(1 \text{ row})$

$$\begin{bmatrix} 1 & 2 & 10 & 100 \\ 2 & 4 & 20 & 200 \\ 1 & 2 & 10 & 100 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 10 & 100 \end{bmatrix} = CR$$

If the column space is a line in 3-dimensional space

then the row space is a line in 4-dimensional space

A adds up (Column k of C) (Row k of R) $=$ **New way to multiply $\boldsymbol{CR}$**

Rank r matrix $=$ Sum of r matrices of rank 1

Geometry of A: Four Fundamental Subspaces

Column space $\mathbf{C}(A)$ = all combinations of columns = all $A\boldsymbol{x}$

Row space $\mathbf{C}(A^{\mathrm{T}})$ = all combinations of columns of A^{T} = all $A^{\mathrm{T}}\boldsymbol{y}$

Nullspace $\mathbf{N}(A)$ = all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$

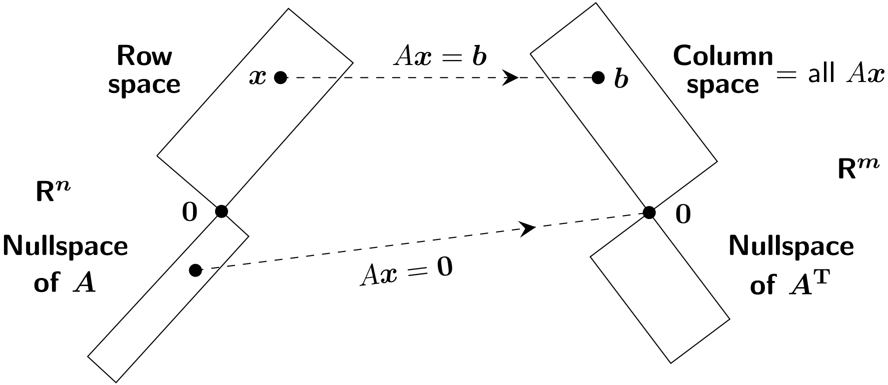
Nullspace of A^{T} $\mathbf{N}(A^{\mathrm{T}})$ = all solutions $\boldsymbol{y}$ to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$

Dimensions $\boldsymbol{r}$ $\boldsymbol{r}$ $\boldsymbol{n-r}$ $\boldsymbol{m-r}$

Row space is orthogonal to nullspace!

$$\begin{bmatrix} \text{row } 1 \\ \cdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdot \\ \mathbf{0} \end{bmatrix}$$

m rows and n columns $\quad$ r independent rows and columns



BIG PICTURE OF LINEAR ALGEBRA

Square invertible matrices $m = n = r$

Nullspaces = zero vector only

Magic factorization $\quad A = CW^{-1}R_*$

$\boldsymbol{C} = r$ independent columns of A $\qquad R_* = r$ independent rows of A

$\boldsymbol{W} = r \times r$ matrix = **intersection of columns in C and rows in R_***

The factorization is just block elimination on A. The block pivot is W.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \end{bmatrix}$$

$\boldsymbol{W}\boldsymbol{R} = \boldsymbol{R}^*$ is exactly r rows of $CR = A$. So W must be invertible

Randomized linear algebra $A \approx CW^{-1}R_*$

Large matrices / thin samples "Skeleton factors"

References to CUR_* R. Penrose (1956) *On best approximate solutions of linear matrix equations,* Math. Proc. Cambridge Phil. Soc. $\mathbf{52}$ 1719.

Hamm and Huang (2020) *Perspectives on CUR Decompositions* arXiv 1907.12668 and ACHA $\mathbf{48}$

Goreinov, Tyrtyshnikov, and Zamarashkin (1997) *Pseudoskeleton approximation* LAA $\mathbf{261}$

Martinsson and Tropp (2020) *Randomized numerical linear algebra : Foundations and Algorithms* Acta Numerica and arXiv : 2002.01387

Randomized Numerical Linear Algebra $A \approx CUR$

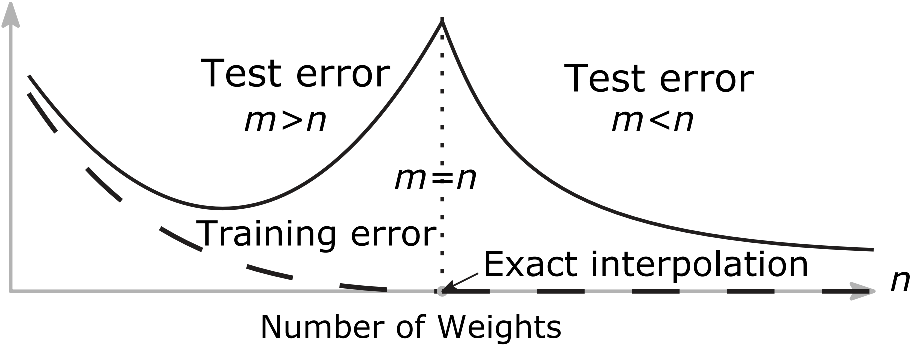
Famous Factorizations of a Matrix

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{LU} && = \text{(lower triangular } L)\,\text{(upper triangular } R)\\
\boldsymbol{A} &= \boldsymbol{QR} && = \text{(orthogonal columns in } Q)\,\text{(upper triangular } R)\\
\boldsymbol{S} &= \boldsymbol{Q\Lambda Q}^{\mathbf{T}} && = \text{(eigenvectors in } Q)\,\text{(eigenvalues in } \Lambda)\\
\boldsymbol{A} &= \boldsymbol{U\Sigma V}^{\mathbf{T}} && = \text{(singular vectors in } U \text{ and } V)\,\text{(singular values in } \Sigma)\\
A\boldsymbol{v}_k &= \sigma_k \boldsymbol{u}_k && \quad \text{(orthogonal vectors } \boldsymbol{v} \text{ mapped to orthogonal vectors } \boldsymbol{u})
\end{aligned}
$$

$$
\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

Full rank $\boldsymbol{r = m = n}$	$\boldsymbol{r = n}$ indep. columns	$\boldsymbol{r = m}$ indep. rows
$\boldsymbol{A}$ is invertible	$\boldsymbol{A^{\mathrm{T}}A}$ is invertible	$\boldsymbol{AA^{\mathrm{T}}}$ is invertible
$\begin{bmatrix} & A & \\ & & \end{bmatrix}$	$\begin{bmatrix} & A & \\ & & \\ & & \end{bmatrix}$	$\begin{bmatrix} & & A & & \\ & & & & \end{bmatrix}$
Solve $A\boldsymbol{x} = \boldsymbol{b}$	$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$	$AA^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{b} \to \overline{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{y}$
$\boldsymbol{x}$ **exact** solution	$\widehat{\boldsymbol{x}}$ **least squares** solution	$\overline{\boldsymbol{x}}$ **minimum norm** solution

The minimum norm solution $\overline{\boldsymbol{x}}$ has no nullspace component / use the pseudoinverse $\overline{\boldsymbol{x}} = A^{+}\boldsymbol{b}$



Deep learning has found that overfitting can help ! A big question in the theory of neural networks using ReLU

Video Lectures	**ocw.mit.edu/courses/mathematics**	**YouTube/mitocw**
Math 18.06	Linear Algebra (including 2020 Vision)	
Math 18.065	Deep Learning	

Books

Introduction to Linear Algebra, (2016)	math.mit.edu/linearalgebra
Linear Algebra & Learning from Data (2019)	math.mit.edu/learningfromdata
Linear Algebra for Everyone (2020)	math.mit.edu/everyone