Bohemian Inner Inverses: A First Step Towards Bohemian Generalized Inverses

J. Rafael Sendra

Departamento de Física y Matemáticas.
Universidad Alcalá

Joint work with: Laureano González–Vega, Juana Sendra, Robert Corless and Eunice Chan

Partially supported by the Spanish Ministerio de Economía y Competitividad, and by the European Regional Development Fund (ERDF), under the Project MTM2017-88796-P.

This talk is framed in the intersection area of **Bohemian matrices** and **Generalized inverses**.

Or maybe I should say: the attempt of the Bohemian foray into the realm of generalized inverses.

Since we all here know about bohemian matrices and we have already heard several talks on the topic, beside few words on bohemian, I will mainly focus, in the first part of the talk, on inner inverses.

$$\mathscr{B}_{\mathbf{m\times n}}(\mathbb{P}) = \{\text{all } m \times n \text{ bohemians with population } \mathbb{P}\}$$

Bohemian=BOunded HEight Matrix of Integers.

From my point of view the hardest difficulty is the fact that the matrix entries are **BO**unded since it turns very complicated to endow $\mathscr{B}_{\mathbf{m\times n}}(\mathbb{P})$ of an algebraic structure. Even defining binary operations entails difficulties.

Of course, one can consider special cases as the group of permutations: that are $\{0, 1\}$–bohemian and have an strong algebraic structure.

Of course, this difficulty is the source of multiple interesting and challenging questions that could be essentially classified in two families:

1. Bound, or give a precise formula for, the height of the [**choose**: characteristic pol, minimal pol, eigenvalue modules, etc] for matrices in $\mathscr{B}_{\mathbf{m}\times\mathbf{n}}(\mathbb{P})$
2. For a given matrix property [**choose**: nilpotency, idempotency, invertibility, etc...] describe, compute, quantify, the subset of $\mathscr{B}_{\mathbf{m}\times\mathbf{n}}(\mathbb{P})$ satisfying this property.

   For the invertibility

   1. ask the inverse to be again bohemian (rhapsodic; Corless) For instance: Mandelbrot matrices are rhapsodic.

      E.Y.S. Chan, R.M. Corless, L. Gonzalez-Vega, J.R. Sendra, J. Sendra. Algebraic linearizations of matrix polynomials. Linear Algebra Appl. 563 (2019) 373–399.

   2. ask the inverse to be similar to a bohemian (S. E. Thornton's PhD Thesis)

There is room enough (plenty of open questions) to work on rhapsodic matrices. **Nevertheless**, we here turn towards a broader point of view, namely generalized inverses, to start this study.

To be more precise, let $\mathbb{K}$ be a field and φ an involutory automorphism over $\mathbb{K}$, that is, $\varphi \circ \varphi$ is the identity; the underlying idea of the involutory automorphism is to have a generalized notion, of the concept of conjugation in $\mathbb{C}$, to be applied over other fields.

Then, for a given $m \times n$ matrix $A = (a_{ij})$, over $\mathbb{K}$, we consider the axioms

$$(1) \quad AXA = A \qquad (2) \quad XAX = X \qquad (3) \quad AX = X^*A^* \qquad (4) \quad XA = A^*X^*$$

where X is an $n \times m$ matrix over $\mathbb{K}$, and where A^* (respectively X^*) denotes the transpose matrix of $\varphi(A) = (\varphi(a_{ij}))$: $A^* = \varphi(A)^T$.

- If X satisfies the four axioms, X is called the **Moore-Penrose inverse**;
- For the **Drazin inverse** (we skip the details here): $A^{\mathrm{Index}(A)+1}X - A^{\mathrm{Index}(A)} = \mathbf{0},\ XAX - X = \mathbf{0},\ AX - XA = \mathbf{0}$.
- For $S \subset \{1, 2, 3, 4\}$, in the literature, the notation $\mathbf{A\{S\}}$ is used to represent the set of all matrices X satisfying the conditions in S.
- Elements in $A\{1\}$ are called **inner inverses**.
- Elements in $A\{2\}$ are called **outer inverses**.
- In addition, one can also analyze generalized inverses with a **prescribed kernel and/or range space**: $A\{S\}_{\mathrm{Ker}(B),*}, A\{S\}_{*,\mathrm{Im}(C)}, A\{S\}_{\mathrm{Ker}(B),\mathrm{Im}(C)}$.

We start our analysis with inner inverses.

Inner inverses are important because of two main reasons

- $A\{1\}$ is geometrically simple. It is an **affine linear space of dimension $\mathbf{mn} - \mathrm{rank}(\mathbf{A})$**, where $A \in \mathbb{K}^{m\times n}$.
- In addition, due to the representation theorem of Urquhart, many generalized inverses can be expressed or represented by means of inner inverses.

Urquhart's Theorem

Let $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times k}$, $C \in \mathbb{K}^{l\times m}$ and $X := B(CAB)^{(1)}C$. Then:

1. $X \in A\{1\}$ if and only if $\mathrm{rank}(CAB) = \mathrm{rank}(A)$.
2. $X \in A\{2\}_{\mathbb{R}(B),\star}$ if and only if $\mathrm{rank}(CAB) = \mathrm{rank}(B)$.
3. $X \in A\{2\}_{\star,\mathbb{N}(C)}$ if and only if $\mathrm{rank}(CAB) = \mathrm{rank}(C)$.
4. $X \in A\{2\}_{\mathbb{R}(B),\mathbb{N}(C)}$ if and only if $\mathrm{rank}(CAB) = \mathrm{rank}(B) = \mathrm{rank}(C)$.
5. $X \in A\{1,2\}_{\mathbb{R}(B),\mathbb{N}(C)}$ if and only if $\mathrm{rank}(CAB) = \mathrm{rank}(B) = \mathrm{rank}(C) = \mathrm{rank}(A)$.

Definition

Let $A \in \mathscr{B}_{m\times n}(\mathbb{P})$ be a bohemian matrix with population $\mathbb{P}$. The elements in

$$\mathbf{(A)}_{\mathscr{B}(\mathbb{P})}\{\mathbf{1}\} := A\{1\} \cap \mathscr{B}_{n\times m}(\mathbb{P})$$

are called **bohemian inner inverses of A**.

Problem Statement

Given

1. a bounded population $\mathbb{P} \subset \mathbb{Z}$, and
2. a bohemian matrix $A \in \mathscr{B}_{m\times n}(\mathbb{P})$,

describe the set $(A)_{\mathscr{B}(\mathbb{P})}\{1\}$ and **compute** $\#((A)_{\mathscr{B}(\mathbb{P})}\{1\})$.

notation

$\mathbf{1}_{r\,s}$ is the $r \times s$ matrix with all entries being 1. Similarly with $-\mathbf{1}_{r\,s}$ and $\mathbf{0}_{r\,s}$:

Definition

We say that $A \in \mathbb{K}^{m\times n}$ is **full** if A has one of the following forms

- Type I: $A = \left(\pm\mathbf{1}_{m\,n} \right)$.
- Type II: $A = \left(\pm\mathbf{1}_{m\,n_1} \mid \mp\mathbf{1}_{m\,n_2} \right)$.
- Type III: $A = \left(\pm\mathbf{1}_{m\,n_1} \mid \mathbf{0}_{m\,n_2} \right)$.
- Type IV: $A = \left(\pm\mathbf{1}_{m\,n_1} \mid \mp\mathbf{1}_{m\,n_2} \mid \mathbf{0}_{m\,n_3} \right)$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

Definition

We say that $A \in \mathbb{K}^{m\times n}$ is **well-settled** if, after multiplying by suitable permutation matrices P, Q, PAQ is of the form

$$PAQ = \left(\begin{array}{c|c|c} M_{p_1q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & M_{p_sq_s} \end{array}\right)$$

where each $M_{p_iq_i}$ is full. We say that a well-settled matrix is **pure** is all matrices $M_{p_iq_i}$ are of the same type, either $\pm\mathbf{1}$ or $\left(\begin{array}{c|c} \pm\mathbf{1} & \mp\mathbf{1} \end{array}\right)$; otherwise, we say that the matrix is **mixed**.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

Definition

We say that $A \in \mathbb{K}^{m\times n}$ is **full** if A has one of the following forms

- Type I: $A = \begin{pmatrix} \pm\mathbf{1}_{m\,n} \end{pmatrix}$.
- Type II: $A = \left(\begin{array}{c|c} \pm\mathbf{1}_{m\,n_1} & \mp\mathbf{1}_{m\,n_2} \end{array}\right)$.
- Type III: $A = \left(\begin{array}{c|c} \pm\mathbf{1}_{m\,n_1} & \mathbf{0}_{m\,n_2} \end{array}\right)$.
- Type IV: $A = \left(\begin{array}{c|c|c} \pm\mathbf{1}_{m\,n_1} & \mp\mathbf{1}_{m\,n_2} & \mathbf{0}_{m\,n_3} \end{array}\right)$.

Definition

We say that $A \in \mathbb{K}^{m\times n}$ is **well-settled** if, after multiplying by suitable permutation matrices P, Q, PAQ is of the form

$$PAQ = \left(\begin{array}{c|c|c} M_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & M_{p_s q_s} \end{array}\right)$$

where each $M_{p_i q_i}$ is full. We say that a well-settled matrix is **pure** is all matrices $M_{p_i q_i}$ are of the same type, either $\pm\mathbf{1}$ or $\left(\begin{array}{c|c} \pm\mathbf{1} & \mp\mathbf{1} \end{array}\right)$; otherwise, we say that the matrix is **mixed**.

Lemma

Let $A \in \mathbb{K}^{m\times n}$, let $P \in \mathbb{K}^{m\times m}$ and $Q \in \mathbb{K}^{n\times n}$ be non-singular matrices, and $\lambda \in \mathbb{K}\setminus\{0\}$.

1. $(A\{1\})^T = A^T\{1\}$.
2. $(\lambda A)\{1\} = \dfrac{1}{\lambda}(A\{1\})$.
3. $Q(PAQ)\{1\}P = A\{1\}$.
4. $\mathbf{0}$ part can be ignored...

$\mathbf{0}$ can be ignored

Lemma

Let $A = \left(\begin{array}{c|c} B_{m\,n_1} & \mathbf{0}_{m\,n_2} \end{array} \right) \in \mathbb{K}^{m\times(n_1+n_2)}$. Then

$$A\{1\} = \left\{ \left(\frac{X_{n_1\ m}}{Y_{n_2\ m}} \right) \middle| X_{n_1\ m} \in B_{m\,n_1}\{1\}, Y_{n_2\ m} \in \mathbb{K}^{n_2\times m}(\mathbb{P}) \right\}$$

The direct brute-force approach is not feasible because the set turns to be huge.

$n \times (n-1)$	Bohemian inners	Bohemians
2×1	2	9
3×2	126	729
4×3	69.576	531.441
5×4	363.985.680	3.486.784.401
6×5	17.812.283.544.870	205.891.132.094.649
7×6	806.9792.560.277.356.314	1 09.418.989.131.512.359.209
8×7	33.609.055.109.399.933.461.665.528	523.347.633.027.360.537.213.511.521

Case of Full Matrices of type I with $\mathbb{P} = \{0, \pm 1\}$

$2n \times (2n-1)$	Bohemian inners	Bohemians
2×4	2	531,441
4×6	2,907	205,891,132,094,649
6×8	363,985,680	523,347,633,027,360,537,213,511,521

Case of Well Settled Matrices, where $\mathbb{P} = \{0, \pm 1\}$ with 2 diagonal blocks of type I.

1. A complete description of the Bohemian inner inverses of full matrices for any population from any field
2. A complete description of the Bohemian inner inverses of well-settled matrices for any population from any field
3. For the population $\mathbb{P} = \{0, \pm 1\}$ we give exact formulas for the number of Bohemian matrices of full and well-settled matrices, respectively

Notation

For $X \in \mathbb{K}^{n\times m}$, we use the notation $\mathbf{S(X)}$ =sum of all the entries of X.

Theorem: Bohemian inner inverses of full matrices

1. Let $\mathbb{P} \subset \mathbb{K}$, such that $1 \in \mathbb{P}$. Then

$$(\mathbf{1_{m\,n}})_{\mathscr{B}(\mathbb{P})}\{\mathbf{1}\} = \{X \in \mathscr{B}_{n\times m}(\mathbb{P}) \,|\, S(X) = 1\}\,.$$

2. Let $\mathbb{P} = \{0, \pm 1\}$ then

$$\#((\mathbf{1_{m\,n}})_{\mathscr{B}(\mathbb{P})}\{\mathbf{1}\}) = \sum_{s=0}^{\lfloor \frac{nm-1}{2} \rfloor} \binom{nm}{s}\binom{nm-s}{s+1}.$$

Theorem: Bohemian inner inverses of full matrices

1. Let $\mathbb{P} \subset \mathbb{K}$, such that $1 \in \mathbb{P}$. Then
$$(\mathbf{1_{m\,n}})_{\mathscr{B}(\mathbb{P})}\{\mathbf{1}\} = \{X \in \mathscr{B}_{n\times m}(\mathbb{P}) \,|\, S(X) = 1\}\,.$$
2. Formula ...

$$\# = 16, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{\{0,\pm 1\}}\{\mathbf{1}\} =$$

$$\left\{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\right.$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\left.\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right\}$$

Theorem: Bohemian inner inverses of full matrices

1. Let $\mathbb{P} \subset \mathbb{K}$, such that $\pm 1 \in \mathbb{P}$. Then

$$\left(\mathbf{1}_{m\,n_1} \;\middle|\; -\mathbf{1}_{m\,n_2} \right)_{\mathscr{B}(\mathbb{P})} \{1\} = \left\{ \begin{pmatrix} X^1 \\ \hline X^2 \end{pmatrix} \in \mathbb{K}^{n \times m} \;\middle|\; \begin{array}{l} X^i \in \mathscr{B}_{n_i \times m}(\mathbb{P}),\ \ i \in \{1,2\}, \\ S(X^1) = 1 + S(X^2) \end{array} \right\}$$

2. Let $\mathbb{P} = \{0, \pm 1\}$ then

$$\#\left(\left(\mathbf{1}_{m\,n_1} \;\middle|\; -\mathbf{1}_{m\,n_2} \right)_{\mathscr{B}(\mathbb{P})} \{1\} \right) =$$

$$\sum_{r_2=0}^{n_2 m} \sum_{s_2=0}^{n_2 m} \sum_{s_1=0}^{\lfloor \frac{(n_1+n_2)m-1}{2} - r_2 \rfloor} \binom{n_1 m}{s_1} \binom{n_2 m}{s_2} \binom{n_2 m - s_2}{r_2} \binom{n_1 m - s_1}{1 + r_2 + s_1 - s_2}.$$

Theorem: Bohemian inner inverses of full matrices

1. Let $\mathbb{P} \subset \mathbb{K}$, such that $\pm 1 \in \mathbb{P}$. Then

$$\left(\mathbf{1}_{m\,n_1} \;\middle|\; -\mathbf{1}_{m\,n_2} \right)_{\mathscr{B}(\mathbb{P})}\{\mathbf{1}\} = \left\{ \left(\frac{X^1}{X^2} \right) \in \mathbb{K}^{n \times m} \;\middle|\; \begin{array}{l} X^i \in \mathscr{B}_{n_i \times m}(\mathbb{P}),\ i \in \{1,2\}, \\ S(X^1) = 1 + S(X^2) \end{array} \right\}$$

2. Formula....

$$\# = 16, \qquad \left(\begin{array}{c|c} 1 & -1 \\ 1 & -1 \end{array} \right)_{\{0,\pm 1\}}\{\mathbf{1}\} =$$

$$\left\{ \left(\frac{-1 \quad 0}{-1 \quad -1} \right), \left(\frac{-1 \quad 1}{-1 \quad 0} \right), \left(\frac{-1 \quad 1}{0 \quad -1} \right), \left(\frac{0 \quad -1}{-1 \quad -1} \right), \left(\frac{0 \quad 0}{-1 \quad 0} \right), \right.$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Definition

Let $X \in \mathbb{K}^{r\times s}$ and let $r_1, r_2 \in \mathbb{Z}$ be such that $r_1 > 0, r_2 \geq 0$ and $r_1 + r_2 = r$. We say that X is **(r_1, r_2)–balanced** if X can be expressed as

$$X = \left(\frac{X^1}{X^2}\right),$$

where $X^i \in \mathbb{K}^{r_i\times s}$, and $S(X^1) = S(X^2)$.
Observe that if X $(r_1, 0)$–balanced means that $S(X) = 0$.

Theorem: Bohemian inner inverses of $\pm\mathbf{1}$-pure well-settled matrices

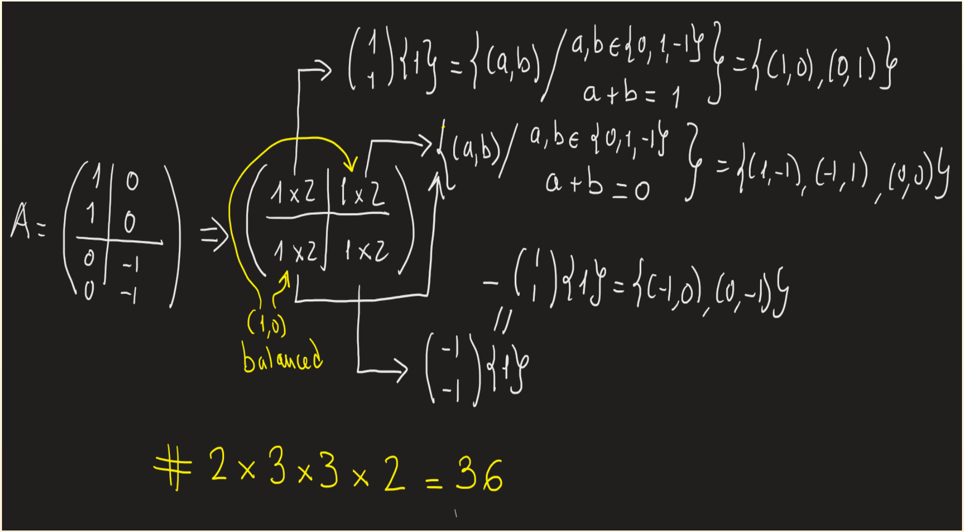
Let $\mathbb{P} \subset \mathbb{K}$, such that $\pm 1 \in \mathbb{P}$, and let

$$A = \left(\begin{array}{c|c|c} \epsilon_1 \mathbf{1}_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \epsilon_s \mathbf{1}_{p_s q_s} \end{array}\right) \in \mathbb{K}^{m\times n}$$

where $\epsilon_k \in \{-1, 1\}$. It holds that

$$\mathbf{A}_{\mathscr{B}(\mathbb{P})}\{1\} = \left\{ \left(\begin{array}{c|c|c} B^{11}_{q_1 p_1} & \cdots & B^{1s}_{q_1, p_s} \\ \hline \vdots & \ddots & \vdots \\ \hline B^{s1}_{q_s p_s} & \cdots & B^{ss}_{q_s p_s} \end{array}\right) \in \mathbb{K}^{n\times m} \;\middle|\; \begin{array}{l} B^{ii}_{q_i p_i} \in (\epsilon_i \mathbf{1}_{p_i q_i})_{\mathscr{B}(\mathbb{P})}\{1\} \\ B^{ij}_{q_i p_j} \text{ is } (q_j, 0)\text{-balanced} \\ \text{if } i \neq j \end{array} \right\}.$$

Bohemian inner inverses of $\pm\mathbf{1}$-pure well settled matrices: a moderate example



Corollary

Let

$$A = \left(\begin{array}{c|c|c} \epsilon_1 \mathbf{1}_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \epsilon_s \mathbf{1}_{p_s q_s} \end{array}\right) \in \mathbb{K}^{m \times n}$$

where $\epsilon_k \in \{-1, 1\}$. **The cardinality of $A_{\mathscr{B}(\{0,\pm 1\})}\{1\}$ is**

$$\left(\prod_{k=1}^{s} \sum_{s=0}^{\lfloor \frac{p_k q_k - 1}{2} \rfloor} \binom{p_k q_k}{s} \binom{p_k q_k - s}{s+1} \right) \left(\prod_{\substack{k_1, k_2 \in \{1, \ldots, s\} \\ k_1 \neq k_2}} \sum_{i=0}^{p_{k_1} q_{k_2}} \binom{p_{k_1} q_{k_2}}{i} \binom{p_{k_1} q_{k_2} - i}{i} \right)$$

Theorem (Bohemian inner inverses of $(\pm\mathbf{1} \;\; \mp\mathbf{1})$-pure well-settled matrices)

Let $\mathbb{P} \subset \mathbb{K}$, such that $\pm 1 \in \mathbb{P}$, let $V_{p_i q_i} = \left(\begin{array}{c|c} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{array} \right)$, where $p_{i1} + q_{i2} = q_i$, and let

$$A = \left(\begin{array}{c|c|c} \epsilon_1 V_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \epsilon_s V_{p_s q_s} \end{array} \right) \in \mathbb{K}^{m \times n},$$

where $\epsilon_k \in \{-1, 1\}$. It holds that

$$A_{\mathbb{P}}\{1\} = \left\{ \left(\begin{array}{c|c|c} B^{11}_{q_1 p_1} & \cdots & B^{1s}_{q_1, p_s} \\ \hline \vdots & \ddots & \vdots \\ \hline B^{s1}_{q_s p_s} & \cdots & B^{ss}_{q_s p_s} \end{array} \right) \in \mathbb{K}^{n \times m} \;\middle|\; \begin{array}{l} B^{ii}_{q_i p_i} \in (\epsilon_i V_{p_i q_i})_{\mathscr{B}(\mathbb{P})}\{1\} \\ B^{ij}_{q_i p_j} \text{ is } (q_{i1}, q_{i2})\text{–balanced if } i \neq j \end{array} \right\}$$

.

Corollary

Let $V_{p_i q_i} = \left(\begin{array}{c|c} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{array} \right)$, where $p_{i1} + q_{i2} = q_i$, and let

$$A = \left(\begin{array}{c|c|c} \epsilon_1 V_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \epsilon_s V_{p_s q_s} \end{array} \right) \in \mathbb{K}^{m \times n},$$

where $\epsilon_k \in \{-1, 1\}$. The cardinality of $A_{\mathscr{B}\{0,\pm\}}\{1\}$ is

$$\left(\prod_{i=1}^{s} \sum_{r_2=0}^{q_{i2}\,p_i} \sum_{s_2=0}^{q_{i2}\,p_i} \sum_{s_1=0}^{\lfloor \frac{(q_{i1}+q_{i2})p_i - 1}{2} - r_2 \rfloor} \binom{q_{i1}\,p_i}{s_1} \binom{q_{i2}\,p_i}{s_2} \binom{q_{i2}\,p_i - s_2}{r_2} \binom{q_{i1}\,p_i - s_1}{1 + r_2 + s_1 - s_2} \right)$$

$$\left(\prod_{\substack{k_1, k_2 \in \{1, \ldots, s\} \\ k_1 \neq k_2}} \sum_{\ell = -t_{1,2} k_2}^{t_{1,2} k_2} \sum_{i=0}^{T_{1,2} k_2} \binom{k_2 T_{1,2}}{i} \binom{k_2 T_{1,2} - i}{\ell + i} \right)$$

where $t_{1,2} = \text{mín}\{q_{k_1 1}, q_{k_1 2}\}$ and $T_{1,2} = \text{máx}\{q_{k_1 1}, q_{k_1 2}\}$.

Theorem (Bohemian inner inverses of mixed well-settled matrices)

Let $\mathbb{P} \subset \mathbb{K}$, such that $\pm 1 \in \mathbb{P}$, and let

$$A = \left(\begin{array}{c|c|c} M_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & M_{p_s q_s} \end{array}\right) \in \mathbb{K}^{m \times n}$$

be a mixed well settled matrix. Then, $A_{\mathscr{B}(\mathbb{P})}\{1\}$ is the set

$$\left\{ \left(\begin{array}{c|c|c} B^{11}_{q_1 p_1} & \cdots & B^{1s}_{q_1, p_s} \\ \hline \vdots & \ddots & \vdots \\ \hline B^{s1}_{q_s p_s} & \cdots & B^{ss}_{q_s p_s} \end{array}\right) \in \mathbb{K}^{n \times m} \;\middle|\; \begin{array}{l} B^{ii}_{q_i p_i} \in (M_{p_i q_i})_{\mathscr{B}(\mathbb{P})}\{1\} \\ B^{ij}_{q_j p_i} \text{ is } \begin{cases} (q_j, 0)\text{–balanced} \\ \text{if } M_{p_i q_i} = \pm \mathbf{1}_{p_i q_i} \\ (q_{i_1}, q_{i_2})\text{–balanced} \\ \text{if } M_{p_i q_i} = \pm \left(\begin{array}{c|c} \mathbf{1}_{p_i q_{i_1}} & -\mathbf{1}_{p_i q_{i_2}} \end{array} \right). \end{cases} \end{array} \right\}.$$

Corollary

Let

$$A = \left(\begin{array}{c|c|c} M_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \hline & \ddots & \\ \mathbf{0} & & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & M_{p_s q_s} \end{array}\right) \in \mathbb{K}^{m\times n}$$

be a mixed well weight matrix. Let $I, J \subset \mathbb{N}$ with $I \cup J = \{1, \ldots, s\}$ and such that $M_{p_i q_i} = \pm \mathbf{1}$ if $i \in I$ and $M_{p_i q_i} = \pm \left(\begin{array}{c|c} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{array} \right)$ if $i \in J$. Then, the **cardinality** of $\mathbf{A}_{\mathscr{B}(\{\mathbf{0},\pm\mathbf{1}\})}\{1\}$ is

$$\left(\prod_{i\in I} \sum_{s=0}^{\lfloor \frac{p_i q_i - 1}{2} \rfloor} \binom{p_i q_i}{s} \binom{p_i q_i - s}{s+1} \right)$$

$$\left(\prod_{i\in J} \sum_{r_2=0}^{q_{i2}\, p_i} \sum_{s_2=0}^{q_{i2}\, p_i} \sum_{s_1=0}^{\lfloor \frac{(q_{i1}+q_{i2})p_i - 1}{2} - r_2 \rfloor} \binom{q_{i1}\, p_i}{s_1} \binom{q_{i2}\, p_i}{s_2} \binom{q_{i2}\, p_i - s_2}{r_2} \binom{q_{i1}\, p_i - s_1}{1 + r_2 + s_1 - s_2} \right)$$

$$\left(\prod_{\substack{k_1 \in I,\, k_2 \in \{1,\ldots,s\} \\ k_1 \neq k_2}} \sum_{i=0}^{p_k q_k} \binom{p_k q_k}{i} \binom{p_k q_k - i}{i} \right)$$

$$\left(\prod_{\substack{k_1 \in J,\, k_2 \in \{1,\ldots,s\} \\ k_1 \neq k_2}} \sum_{\ell=-t_{1,2}k_2}^{t_{1,2}k_2} \sum_{i=0}^{T_{1,2}k_2} \binom{k_2 T_{1,2}}{i} \binom{k_2 T_{1,2} - i}{\ell + i} \right)$$

where $t_{1,2} = \text{mín}\{q_{k_1 1}, q_{k_1 2}\}$ and $T_{1,2} = \text{máx}\{q_{k_1 1}, q_{k_1 2}\}$.