Recent Progress in the Rational Factorisation of Integer Matrices

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The Approach of Louis J. Mordell (1888-1972) I

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Mordell considered the similarity classes of $n \times n$ symmetric matrices with determinant 1.

He considered two such matrices L, M to be in the same class if there exists a unimodular integral matrix N such that $M = N^T LN$.

The number of such similarity classes is denoted by h_n.

Hence a matrix M is in the similarity class of I_n (the identity matrix) if and only if there exists a factorisation $M = N^T N$ with N an integer matrix N.

The Approach of Louis J. Mordell II

The quadratic form classically associated with the symmetric matrix M, $q(x) = x^T Mx$, can be written as

$$q(x) = x^T Mx = x^T N^T Nx = y^T y = \sum_{j=1}^{n} y_j^2, \tag{1}$$

where $y = Nx$, and N has determinant 1. Thus, the factorisation can be used to write the quadratic form $q(x)$ as a sum of squares of n linear factors.

When $n = 8$, such a factorisation may not exist.

Minkowski proved in 1911 that $h_n \geq [1 + n/8]$, so $h_n \geq 2$ if $n = 8$. Mordell showed (1938) that $h_8 = 2$, and Ko showed (1938) that $h_9 = 2$ as well.

Talk Focus

In this talk, we revisit the question of integer matrix factorisation in the light of recent general results on matrix decompositions.

We establish that the existence of integer solutions to a certain quadratic equation is a necessary condition for a matrix factorisation of the type $M = N^T N$ (for symmetric positive definite M) to exist.

It is interesting to note that solutions to this new type of quadratic equation associated with a given integer matrix M can also lead to rational matrix factors N with entries in $\frac{1}{n^2}\mathbb{Z}$.

The Wilson Matrix I

Throughout this talk we use the classical example of the Wilson matrix

$$W = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix} \tag{2}$$

This integer matrix has determinant 1 and hence an integer inverse matrix, but is moderately ill conditioned, despite its small size and entries. It has the integer factorisation $W = Z^T Z$ discovered by the first two authors with

$$Z = \begin{pmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{3}$$

We note that the entries of Z are nonnegative and, although the matrix is not triangular, it has a block upper triangular structure and can be thought of as a block Cholesky factor of W.

The Wilson Matrix II

The quadratic form associated with the Wilson matrix can be written, using (1) and (3), as a sum of four squares:

$$q(x) = x^T Wx = (2x_1 + 3x_2 + 2x_3 + 2x_4)^2 + (x_1 + x_2 + 2x_3 + x_4)^2 + (x_3 + 2x_4)^2 + (x_3 + x_4)^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2. \quad (4)$$

As Z is a unimodular integer matrix it has an integer inverse.

It follows by Lagrange's four-square theorem that the quadratic form q generated by the Wilson matrix is universal in the sense that it generates all positive integers as x ranges over $\mathbb{Z}^4$.

Matrix Decompositions I

The following symmetries of $n \times n$ matrices were considered by the second two authors.

(S) A matrix $M = (m_{i,j})_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ has the *constant sum property* (or is of type S) if there is a number $w \in \mathbb{R}$, called the *weight* of the matrix, such that

$$\sum_{j=1}^{n} m_{i,j} = \sum_{j=1}^{n} m_{j,i} = nw \qquad (i \in \{1, \ldots, n\}).$$

The vector subspace of $\mathbb{R}^{n\times n}$ of matrices having the constant sum property with some weight is denoted by S_n and can be characterised as

$$S_n = \left\{ M \in \mathbb{R}^{n\times n} : 1_n^T M u = 0 = u^T M 1_n \; (u \in \{1_n\}^{\perp}) \right\},$$

where $1_n \in \mathbb{R}^n$ is the column vector with all entries equal to 1 and orthogonality is with respect to the standard inner product, $\{1_n\}^{\perp} = \{u \in \mathbb{R}^n : u^T 1_n = 0\}$.

Matrix Decompositions II

(V) A matrix $M = (m_{i,j})_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$ has the *vertex cross sum property* (or is of type V) if

$$m_{i,j} + m_{k,l} = m_{i,l} + m_{k,j} \quad (i, j, k, l \in \{1, \ldots, n\})$$

and the matrix entries sum to zero, $\sum_{i,j=1}^{n} m_{i,j} = 0$. The vector subspace of $\mathbb{R}^{n \times n}$ of matrices having the vertex cross sum property is denoted by V_n and can be characterised as

$$V_n = \left\{ M \in \mathbb{R}^{n \times n} : u^T M v = 0 \; (u, v \in \{1_n\}^{\perp}), \; 1_n^T M 1_n = 0 \right\} \tag{5}$$

Matrix Decompositions III

The decompositions are unique and the odd and even parts of the decomposition orthogonal w.r.t. the Frobenius norm.

For the integer factor Z of the Wilson matrix we find that the $S \oplus V$ decomposition $Z = Z_V + Z_S + w_Z \mathcal{E}_4$ is given by

$$\begin{aligned} Z &= Z_V + Z_S + w_Z \mathcal{E}_4 \\ &= \frac{1}{8}\begin{pmatrix} 5 & 7 & 11 & 11 \\ -3 & -1 & 3 & 3 \\ -7 & -5 & -1 & -1 \\ -9 & -7 & -3 & -3 \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 3 & 15 & -9 & -9 \\ 3 & -1 & 7 & -9 \\ -5 & -9 & -1 & 15 \\ -1 & -5 & 3 & 3 \end{pmatrix} + \frac{19}{16}\mathcal{E}_4, \end{aligned}$$

so that Z has weight $w_Z = \frac{19}{16}$ and, the entries of Z_V and Z_S are in $\frac{1}{16}\mathbb{Z}$.

Theorem 1

A matrix $M \in \mathbb{R}^{n\times n}$ is an element of V_n if and only if there exist vectors $a, b \in \{1_n\}^\perp$ such that $M = a1_n^T + 1_n b^T$.

Main Results I

For the Wilson matrix W this decomposition takes the form

$$16W = 1_4 \begin{pmatrix} -27 \\ 9 \\ 13 \\ 5 \end{pmatrix}^T + \begin{pmatrix} -27 \\ 9 \\ 13 \\ 5 \end{pmatrix} 1_4^T + \begin{pmatrix} 15 & 11 & -9 & -17 \\ 11 & 23 & -13 & -21 \\ -9 & -13 & 15 & 7 \\ -17 & -21 & 7 & 31 \end{pmatrix} + 119\mathcal{E}_4$$

Theorem 2

Let $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$. Then there is a unique decomposition

$$M = M_V + M_0 + (\mathrm{wt}\, M)\, \mathcal{E}_n,$$

where $M_0 \in S_n$ with weight 0 and $M_V = a1_n^T + 1_n b^T \in V_n$, and the entries of the vectors a and b are given by

$$a_i = \frac{1}{n} \sum_{j=1}^{n} m_{i,j} - \mathrm{wt}\, M \quad (i \in \{1, \dots, n\}),$$

$$b_j = \frac{1}{n} \sum_{i=1}^{n} m_{i,j} - \mathrm{wt}\, M \quad (j \in \{1, \dots, n\}).$$

In particular, if M is an integer matrix then the vectors $n^2 a$ and $n^2 b$ have integer entries.

Main Results II

Theorem 3

Given a symmetric matrix $M \in \mathbb{Z}^{n\times n}$, it is necessary for the existence of a factorisation $M = N^T N$ with $N \in \mathbb{Z}^{n\times n}$ that $n^2 \,\mathrm{wt}\, N \in \mathbb{Z}$ and that the vector components $n^2 a_j, n^2 b_j \in \mathbb{Z}$ $(j \in \{1, \dots, n\})$, where $N = a1_n^T + 1_n b^T + N_0 + w_N \mathcal{E}_n$ is the decomposition of N, form a solution of the quadratic equation

$$n^4 \,\mathrm{wt}\, M = n\left(n^2 \,\mathrm{wt}\, N\right)^2 + \sum_{j=1}^{n} (n^2 a_j)^2.$$

Theorem 4

Using the decomposition $M = y1_n^T + 1_n y^T + M_0 + (\mathrm{wt}\, M)\,\mathcal{E}_n$, we have $b = \frac{1}{nw_N}\left(y - N_0^T a\right)$. Hence matrix factors correspond to solutions N_0 of the quadratic matrix equation

$$N_0^T (aa^T + nw_N^2 \mathcal{I}_n) N_0 - N_0^T a y^T - y a^T N_0 = nw_N^2 M_0 - yy^T. \qquad (6)$$

Theory into Practice I

The quadratic equation arising from balancing the weights in the assumed factorisation of the Wilson matrix $W = Z^T Z$ is

$$\begin{aligned}\mathrm{wt}\, W = \frac{119}{16} &= a_1^2 + a_2^2 + a_3^2 + a_4^2 + 4w_Z^2 \\ &= 2(a_1^2 + a_2^2 + a_3^2 + a_1 a_2 + a_1 a_3 + a_2 a_3) + 4\, w_Z^2,\end{aligned}$$

as $a_4 = -a_1 - a_2 - a_3$. Clearing fractions and setting $x_i = 16a_i$, $w = 16w_Z$, we find that a necessary condition for the integer factorisation of the Wilson matrix is that there are integer solutions to the quadratic equation

$$2w^2 + x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 = 952.$$

Solving this equation for w, x_1, x_2, x_3 in Mathematica 11.0 on a PC with a Intel Core i7 6500CPU, gave the 1728 solutions in just under 6 seconds.

Theory into Practice II

Exactly one third (576) of these solutions lead to rational matrix factorisations $W = Z^T Z$ with $Z \in \frac{1}{16}\mathbb{Z}^{4\times 4}$.

The process of converting solutions into matrix factors, i.e. of finding suitable vectors b and matrices N_0 satisfying the equations of Theorem 4, took considerably longer at 34 minutes.

The approach involved utilising (6), in which the vector b is eliminated and the right-hand-side completely determined for a given factor weight w_N.

Substituting potential solutions for the vector a and weight w_N thus reduces the general problem of finding the matrix N_0 to that of an $(n-1) \times (n-1)$ unknown matrix.

Theory into Practice III

For the Wilson matrix the 576 matrix factors split naturally into 3 classes, where any two elements in the same class are related by an integer orthogonal matrix.

If $M = N^T N$ and U is an integer orthogonal matrix, then UN is another solution of the factorisation problem

Conversely, if $\det M = 1$ and N and N' are solutions with integer entries, then $N' = UN$, where $U = N' N^{-1}$ is an integer orthogonal matrix.

It can be shown that any integer orthogonal matrix is a signed permutation matrix, i.e. a matrix which has exactly one non-zero entry, either 1 or -1, in each row and in each column.

Theory into Practice IV

For the Wilson matrix the 3 classes of matrix factors with entries in $\frac{1}{16}\mathbb{Z}$, can be represented by the integer matrix factor Z and the two rational matrix factors

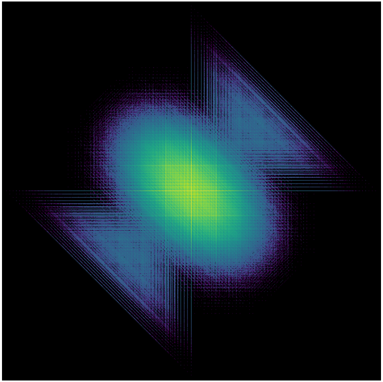
$$Z' = \begin{pmatrix} \frac{1}{2} & 1 & 0 & 1 \\ \frac{3}{2} & 2 & 3 & 3 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 2 & 1 & 0 \end{pmatrix}, \quad Z'' = \begin{pmatrix} \frac{3}{2} & 2 & 2 & 2 \\ \frac{3}{2} & 2 & 2 & 1 \\ \frac{1}{2} & 1 & 1 & 2 \\ -\frac{1}{2} & -1 & 1 & 1 \end{pmatrix}, \qquad (7)$$

where any two elements in the same class are related by an integer orthogonal matrix.

The three factorisations $W = Z^T Z = Z'^T Z' = Z''^T Z''$ correspond to the solutions $(w, x_1, x_2, x_3) = (19, 17, 1, -7)$, $(w, x_1, x_2, x_3) = (18, -8, 20, -12)$ and $(w, x_1, x_2, x_3) = (19, 11, 7, -1)$, respectively.

Note that Z' and Z'' are not integer matrices, so the equivalence class of Z comprises all integer factorisations of the Wilson matrix.

Thank you for listening!



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