Determinants of upper Hessenberg Bohemians

joint work with

Jishe Feng • Gian Maria Negri Porzio

Massimiliano Fasi

Örebro Universitet

Upper Hessenberg matrices

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

Upper Hessenberg

Upper Hessenberg matrices

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times & \times & \times \\ 1 & \times & \times & \times & \times \\ & 1 & \times & \times & \times \\ & & 1 & \times & \times \\ & & & 1 & \times \end{bmatrix}$$

Upper Hessenberg

Normalized Upper Hessenberg

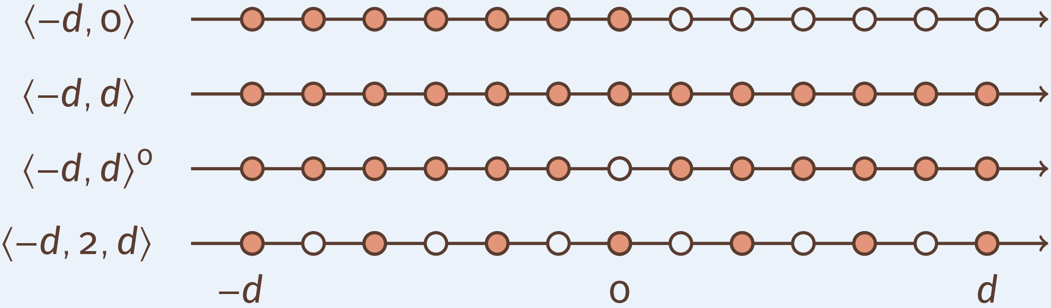
Upper Hessenberg matrices

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times & \times & \times \\ 1 & \times & \times & \times & \times \\ & 1 & \times & \times & \times \\ & & 1 & \times & \times \\ & & & 1 & \times \end{bmatrix} \qquad \begin{bmatrix} 0 & \times & \times & \times & \times \\ 1 & 0 & \times & \times & \times \\ & 1 & 0 & \times & \times \\ & & 1 & 0 & \times \\ & & & 1 & 0 \end{bmatrix}$$

Upper Hessenberg | Normalized Upper Hessenberg | (Normalized) Hollow Upper Hessenberg

Bohemian domains

BOHEMIAN: **BO**unded **HE**ight **M**atrices of **I**ntegers



R. M. Corless, S. E. Thornton. The Bohemian Eigenvalue Project. *ACM Commun. Comput. Algebra*, 50:158–160, 2017.

What we want to know

Pick a “class”

- a matrix structure $\mathcal{H}$
- a size n
- a domain $\mathcal{D}$

Questions

M. What is the largest possible determinant (in absolute value)?

C. How many distinct determinants are there?

Characteristic Polynomial Database

The **characteristic polynomial database** contains characteristic polynomials, minimal polynomials and properties for a variety of families of Bohemian matrices. Currently the database contains **1,762,728,065** characteristic polynomials from **2,366,960,967,336** matrices.

Bohemian Families

Unstructured

Matrices with no structure. For example, 5×5 matrices with entries from the set $\{-1, 0, +1\}$.

Upper Hessenberg

Upper Hessenberg matrices with subdiagonal entries fixed at 1.

Data Files

Data files have been temporarily removed due to unexpected costs, if you would like access to the characteristic polynomial data files please contact me at sthornt7@uwo.ca.

Polynomial Files

All characteristic polynomial and minimal polynomial files are CSV files with no header. They have been provided in uncompressed format, as well as compressed as zip files and tar.gz files. The characteristic polynomial files follow the naming convention `CharPolys_nxn.csv` where `n` is the dimension of the matrices in the family they are computed from. Similarly, the minimal polynomial files follow the naming convention `MinPolys_nxn.csv` where `n` is the dimension of the matrices in the family they are computed from. Each row in the CSV files is of the form:

```
count, c_{n}, c_{n-1}, ..., c_0
```

Conjectures

While computing properties for the families of matrices available in the database, we came across many cases where a property appears to match a sequence on the OEIS but we were unable to find a proof. Below we list sequences that appear to match, for which we have no proof of their validity. If you prove any of the conjectures below, or know of a reference with their proof, please send an email to Steven Thornton at sthornt7@uwo.ca, or Rob Corless at rcorless@uwo.ca with details. Full credit will be given.

Conjecture 1

- **Conjecture:** The number of nilpotent $n \times n$ matrices with entries from the set $\{0, +1\}$ is given by the sequence A003024.
- **Status:** True
- **Reference:** Cvetković, D., Doob, M., & Sachs, H. (1980). Spectra of Graphs-Theory and Application. Third ed., Barth, Heidelber, 1995.
- **Proof:** [p. 81] show that a digraph G contains no cycle if and only if all eigenvalues of the adjacency matrix are 0, which is the explicit bijection between nilpotent $n \times n$ matrices and DAGs.
- **Proof Provided By:** Jianxiang Chen

Conjecture 2

- **Conjecture:** The maximal characteristic height of $n \times n$ matrices with entries from the set $\{0, +1\}$ is given by the sequence A082914.
- **Status:** False
- **Proof:** According to the conjecture, the largest characteristic height of a 20×20 $\{0, 1\}$ matrix should be 9754214400. But observe that the j-th coefficient of the characteristic polynomial is an alternate sum of all the determinants of the $(n - j) \times (n - j)$ diagonal minors, and the determinant of a $\{0, 1\}$ matrix is bounded by A003432, the coefficient is bounded by $C(n, j)$A003432$(n - j)$. The values of $C(n, j)$A003432$(n - j)$ for a 20×20 matrix are: 390625000, 648273920, 1270087680, 1587609600, 2032140288, 988961400, 734657040, 459160650, 244885680, 59121920, 24186240, 7054320, 2480640, 348840, 77520, 14535, 2280, 190, 20, which are all smaller than 9754214400. Therefore, the conjecture is false.
- **Proof Provided By:** Jianxiang Chen

Conjecture 3

- **Conjecture:** The number of nilpotent $n \times n$ matrices with entries from the set $\{0, +1\}$ and diagonal entries fixed at 0 is given by the sequence A003024.

The Characteristic Polynomial Database

Twenty conjectures (for $d = 1$) regarding:

- nilpotency
- characteristic polynomials
- determinants

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

S. E. Thornton, The Characteristic Polynomial Database, 2018.

The CPDB conjectures in 2018

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

- nilpotency
- characteristic polynomials
- determinants

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

- true
- false
- open

We proved

- a generalization of Conjectures 4 and 12–20
- Conjecture 8

A formula for the determinants

Determinant of normalized Hessenberg matrices

For an $n \times n$ normalized upper Hessenberg matrix H

$$\det H = (-1)^{n+1}\left(h_{1n} + \sum_{i=2}^{n} (-1)^{i-1} h_{in} \det H^{(i-1)}\right)$$

where $H^{(i)}$ is the leading principal minor of size i.

Proof. Laplace expansion along the last column.

A formula for the determinants—illustrated

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} \\ 1 & h_{22} & h_{23} & h_{24} & h_{25} \\ & 1 & h_{33} & h_{34} & h_{35} \\ & & 1 & h_{44} & h_{45} \\ & & & 1 & h_{55} \end{bmatrix}$$

$\det H =$

A formula for the determinants—illustrated

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & \boxed{h_{15}} \\ 1 & h_{22} & h_{23} & h_{24} & h_{25} \\ & 1 & h_{33} & h_{34} & h_{35} \\ & & 1 & h_{44} & h_{45} \\ & & & 1 & h_{55} \end{bmatrix}$$

$\det H = h_{15}$

A formula for the determinants—illustrated

$$H = \begin{bmatrix} \boxed{h_{11}} & h_{12} & h_{13} & h_{14} & h_{15} \\ 1 & h_{22} & h_{23} & h_{24} & \boxed{h_{25}} \\ & 1 & h_{33} & h_{34} & h_{35} \\ & & 1 & h_{44} & h_{45} \\ & & & 1 & h_{55} \end{bmatrix}$$

$$\det H = h_{15} - h_{25} \det H^{(1)}$$

A formula for the determinants—illustrated

$$H = \begin{bmatrix} \boxed{\begin{matrix} h_{11} & h_{12} \\ 1 & h_{22} \end{matrix}} & \begin{matrix} h_{13} \\ h_{23} \end{matrix} & \begin{matrix} h_{14} \\ h_{24} \end{matrix} & \begin{matrix} h_{15} \\ h_{25} \end{matrix} \\ \begin{matrix} & 1 \end{matrix} & h_{33} & h_{34} & \boxed{h_{35}} \\ & 1 & h_{44} & h_{45} \\ & & 1 & h_{55} \end{bmatrix}$$

$$\det H = h_{15} - h_{25} \, \det H^{(1)} + h_{35} \, \det H^{(2)}$$

A formula for the determinants—illustrated

$$H = \begin{bmatrix} \boxed{\begin{matrix} h_{11} & h_{12} & h_{13} \\ 1 & h_{22} & h_{23} \\ & 1 & h_{33} \end{matrix}} & \begin{matrix} h_{14} \\ h_{24} \\ h_{34} \end{matrix} & \begin{matrix} h_{15} \\ h_{25} \\ h_{35} \end{matrix} \\ \begin{matrix} & & 1 \\ & & \end{matrix} & \begin{matrix} h_{44} \\ 1 \end{matrix} & \begin{matrix} \boxed{h_{45}} \\ h_{55} \end{matrix} \end{bmatrix}$$

$$\det H = h_{15} - h_{25} \det H^{(1)} + h_{35} \det H^{(2)} - h_{45} \det H^{(3)}$$

A formula for the determinants—illustrated

$$H = \begin{bmatrix} \boxed{\begin{matrix} h_{11} & h_{12} & h_{13} & h_{14} \\ 1 & h_{22} & h_{23} & h_{24} \\ & 1 & h_{33} & h_{34} \\ & & 1 & h_{44} \end{matrix}} & \begin{matrix} h_{15} \\ h_{25} \\ h_{35} \\ h_{45} \end{matrix} \\ \begin{matrix} & & & 1 \end{matrix} & \boxed{h_{55}} \end{bmatrix}$$

$$\det H = h_{15} - h_{25} \det H^{(1)} + h_{35} \det H^{(2)} - h_{45} \det H^{(3)} + h_{55} \det H^{(4)}$$

Constructive proofs are better

Expected answers

M. What is the largest possible determinant (in absolute value)?

Exhibit a matrix with maximum determinant.

C. How many distinct determinants are there?

Construct a matrix with given determinant.

Ideal test matrices

- Only integer arithmetic
- Answer known exactly

Maximum determinant $(\langle -d, 0\rangle, \langle -d, d\rangle, \langle -d, d\rangle^0)$

The maximum determinant is achieved by

$$K^{(d,n)} = \begin{bmatrix} -d & -d & \cdots & -d & -d \\ 1 & -d & \cdots & -d & -d \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & -d & -d \\ & & & 1 & -d \end{bmatrix}$$

which has determinant

$$\det K^{(d,n)} = (-1)^n d(d+1)^{n-1}$$

Proof: By induction.

Maximum determinant $(\langle -d,0\rangle, \langle -d,d\rangle, \langle -d,d\rangle^{0})$

The maximum determinant is achieved by

$$K^{(d,n)} = \left[\begin{array}{cccc|c} -d & -d & \cdots & -d & -d \\ 1 & -d & \cdots & -d & -d \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & -d & -d \\ \hline & & & 1 & -d \end{array}\right] = \left[\begin{array}{c|c} & -d \\ K^{(d,n-1)} & -d \\ & \vdots \\ & -d \\ \hline 1 & -d \end{array}\right]$$

which has determinant

$$\det K^{(d,n)} = (-1)^n d(d+1)^{n-1}$$

Proof: By induction.

Enumerating determinants

The matrix

$$H = \left[\begin{array}{cccc|c} & & & & b \\ & & K^{(d,n-1)} & & a_0 \\ & & & & \vdots \\ & & & & a_{n-3} \\ \hline 0 & \cdots & 0 & 1 & a_{n-2} \end{array}\right] \qquad b, a_0, \ldots, a_{n-2} \in \mathcal{D}$$

Enumerating determinants

The matrix

$$H = \left[\begin{array}{cccc|c} & & & & b \\ & & K^{(d,n-1)} & & a_0 \\ & & & & \vdots \\ & & & & a_{n-3} \\ \hline 0 & \cdots & 0 & 1 & a_{n-2} \end{array}\right] \qquad b, a_0, \ldots, a_{n-2} \in \mathcal{D}$$

has determinant

$$(-1)^{n+1} b + \sum_{i=0}^{n-2} (-1)^{n+i} a_i \det K^{(d,i+1)}$$

Enumerating determinants

The matrix

$$H = \left[\begin{array}{cccc|c} & & & & b \\ & & & & a_0 \\ & K^{(d,n-1)} & & & \vdots \\ & & & & a_{n-3} \\ \hline 0 & \cdots & 0 & 1 & a_{n-2} \end{array}\right]$$

$$b, a_0, \ldots, a_{n-2} \in \mathcal{D}$$
$$\widetilde{b}, \widetilde{a}_0, \ldots, \widetilde{a}_{n-2} \in \mathcal{D}$$

has determinant

$$(-1)^{n+1} b + \sum_{i=0}^{n-2} (-1)^{n+i} a_i \det K^{(d,i+1)} = (-1)^n \left(\widetilde{b} + \sum_{i=0}^{n-2} \widetilde{a}_i \cdot d(d+1)^i \right)$$

Integer representations

$$\gamma = (-1)^n\left(b + \sum_{i=0}^{n-2} a_i \cdot d(d+1)^i\right), \qquad b, a_0, \ldots, a_{n-2} \in \mathcal{D}$$

What integers can we write in this form?

Integer representations

$$\gamma = (-1)^n \left(b + \sum_{i=0}^{n-2} a_i \cdot d(d+1)^i \right), \qquad b, a_0, \ldots, a_{n-2} \in \mathcal{D}$$

What integers can we write in this form?

$$\begin{aligned}
\mathcal{D} &= \langle -d, 0 \rangle & \gamma &\in (-1)^n \cdot \langle 0, d(d+1)^{n-1} \rangle \\
\mathcal{D} &= \langle -d, d \rangle & \gamma &\in \langle -d(d+1)^{n-1}, d(d+1)^{n-1} \rangle \\
\mathcal{D} &= \langle -d, d \rangle^0, d > 1 & \gamma &\in \langle -d(d+1)^{n-1}, d(d+1)^{n-1} \rangle \\
\mathcal{D} &= \langle -1, 1 \rangle^0 & \gamma &\in \langle -2^{n-2}, 2, 2^{n-2} \rangle
\end{aligned}$$

Normalized upper Hessenberg: summary

Answers

M. What is the largest possible determinant (in absolute value)?

$g_{n-1}^{d} := d(d+1)^{n-1}$ regardless of the domain.

C. How many distinct determinants are there?

It depends on the domain.

Constructive proofs

- We can construct a matrix that has given determinant
- All these matrices share the first $n-1$ columns

Hollow normalized Bohemians

Similar ideas

- Maximality first
- Expansion along last column
- Integer representations

Very different results

$$\begin{bmatrix} 0 & \times & \times & \times & \times \\ 1 & 0 & \times & \times & \times \\ & 1 & 0 & \times & \times \\ & & 1 & 0 & \times \\ & & & 1 & 0 \end{bmatrix}$$

(Normalized) Hollow
Upper Hessenberg

A generalization of Fibonacci numbers

d-weighted Fibonacci

$$f_i^d = \begin{cases} 0, & i = 0, \\ d, & i = 1, \\ f_{i-1}^d + d \cdot f_{i-2}^d, & i > 1. \end{cases} \qquad d \in \mathbb{N}.$$

$d = 1$	0	1	1	2	3	5	8	13	21	…
$d = 2$	0	2	2	6	10	22	42	86	170	…
$d = 3$	0	3	3	12	21	57	120	291	651	…

Maximum determinant $(\langle -d, 0\rangle, \langle -d, d\rangle, \langle -d, d\rangle^0)$

The maximum determinant is achieved by

$$\widehat{K}^{(d,n)} = \begin{bmatrix} 0 & -d & \cdots & -d & -d \\ 1 & 0 & \cdots & -d & -d \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & -d \\ & & & 1 & 0 \end{bmatrix}$$

which has determinant

$$\det \widehat{K}^{(d,n)} = (-1)^n \cdot f^d_{n-1}$$

Proof: By induction.

Maximum determinant $(\langle -d,0\rangle, \langle -d,d\rangle, \langle -d,d\rangle^0)$

The maximum determinant is achieved by

$$\widehat{K}^{(d,n)} = \left[\begin{array}{cccc|c} 0 & -d & \cdots & -d & -d \\ 1 & 0 & \cdots & -d & -d \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & -d \\ \hline & & & 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} & -d \\ \widehat{K}^{(d,n-1)} & -d \\ & \vdots \\ & -d \\ \hline 1 & 0 \end{array}\right]$$

which has determinant

$$\det \widehat{K}^{(d,n)} = (-1)^n \cdot f^d_{n-1}$$

Proof: By induction.

Enumerating determinants

The matrix

$$H = \left[\begin{array}{cccc|c} & & & & b \\ & \widehat{K}^{(d,n-1)} & & & a_0 \\ & & & & \vdots \\ & & & & a_{n-3} \\ \hline 0 & \cdots & 0 & 1 & 0 \end{array}\right] \qquad b, a_0, \ldots, a_{n-3} \in \mathcal{D}$$

Enumerating determinants

The matrix

$$H = \left[\begin{array}{cccc|c} & & & & b \\ & \widehat{K}^{(d,n-1)} & & & a_0 \\ & & & & \vdots \\ & & & & a_{n-3} \\ \hline 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

$$b, a_0, \ldots, a_{n-3} \in \mathcal{D}$$
$$\widetilde{b}, \widetilde{a}_0, \ldots, \widetilde{a}_{n-3} \in \mathcal{D}$$

has determinant

$$\det H = (-1)^{n+1} b + \sum_{i=0}^{n-3} (-1)^{n+i} a_i \det \widehat{K}^{(d,i+1)} = (-1)^n \left(\widetilde{b} + \sum_{i=0}^{n-3} \widetilde{a}_i \cdot f_i^d \right)$$

Integer representations

$$\gamma = (-1)^n \left(b + \sum_{i=0}^{n-3} a_i \cdot f_i^d \right), \qquad b, a_0, \ldots, a_{n-3} \in \mathcal{D}$$

What integers can we write in this form?

Integer representations

$$\gamma = (-1)^n \left(b + \sum_{i=0}^{n-3} a_i \cdot f_i^d \right), \qquad b, a_0, \ldots, a_{n-3} \in \mathcal{D}$$

What integers can we write in this form?

$$\mathcal{D} = \langle -d, 0 \rangle \qquad \gamma \in (-1)^n \cdot \langle 0, f_{n-1}^d \rangle$$

$$\mathcal{D} = \langle -d, d \rangle \qquad \gamma \in \langle -f_{n-1}^d, f_{n-1}^d \rangle$$

$$\mathcal{D} = \langle -d, d \rangle^0, d > 1 \qquad \gamma \in \langle -f_{n-1}^d, f_{n-1}^d \rangle$$

$$\mathcal{D} = \langle -1, 1 \rangle^0 \qquad \gamma \in \langle -f_{n-1}, 2, f_{n-1} \rangle$$

Comparison

Convenient notation: $g_n^d = d(d+1)^n$

Comparison

Convenient notation: $g_n^d = d(d+1)^n$

$\mathcal{H}$ \ $\mathcal{D}$	$\langle -d, 0\rangle$	$\langle -d, d\rangle$	$\langle -d, d\rangle^0$	
			$d > 1$	$d = 1$
Normalized	$(-1)^n\langle 0, g_{n-1}^d\rangle$	$\langle -g_{n-1}^d, g_{n-1}^d\rangle$	$\langle -g_{n-1}^d, g_{n-1}^d\rangle$	$\langle -g_{n-1}^1, 2, g_{n-1}^1\rangle$
Hollow	$(-1)^n\langle 0, f_{n-1}^d\rangle$	$\langle -f_{n-1}^d, f_{n-1}^d\rangle$	$\langle -f_{n-1}^d, f_{n-1}^d\rangle$	$\langle -f_{n-1}^1, 2, f_{n-1}^1\rangle$

- All conjectures were true!
- Similar ideas
- Unexpected parallelism in the result

The CPDB conjectures in 2021

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

- nilpotency
- characteristic polynomials
- determinants

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

- true for any $d \in \mathbb{N} \setminus \{0\}$
- true as stated
- false as stated
- open

C8. Generalized proof due to *Keating and Keleş (2020)*

C9. Counterexample due to *Du, da Fonseca, Xu, and Ye (2021)*

References

- M. Fasi and G. M. Negri Porzio. Determinants of normalized Bohemian upper Hessenberg matrices. *Electron. J. Linear Algebra*, 36:352–366, 2020.
- J. P. Keating and A. A. Keleş. Maximum absolute determinants of upper Hessenberg Bohemian matrices. Preprint arXiv:2003.00454 [cs.SC], 2020.
- M. Fasi, J. Feng, and G. M. Negri Porzio. Corrigendum to “Determinants of normalized Bohemian upper Hessenberg matrices” [Electron. J. of Linear Algebra 36 (2020) 352–366]. *Electron. J. Linear Algebra*, 37:160–162, 2021.
- Z. Du, C. da Fonseca, Y. Xu, and J. Ye. Disproving a conjecture of Thornton on Bohemian matrices. *Open Mathematics*, 19(1):505–514, 2021.