Experiments on Upper Hessenberg and Toeplitz Bohemians

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Bohemian Matrices and Applications Minisymposium
SIAM Annual Meeting
July 22, 2021

Joint work with Robert M. Corless, Laureano Gonzalez-Vega, J. Rafael Sendra, Juana Sendra, & Steven E. Thornton

Introduction

A family of **Bohemians** (**Bo**unded **He**ight **M**atrix of **I**ntegers) is a distribution of random matrices where the matrix entries are sampled from a discrete set of bounded height.

In this talk, we will be looking at specific matrix structures of Bohemians:

1. Upper Hessenberg Bohemians
2. Upper Hessenberg Toeplitz Bohemians

Upper Hessenberg Bohemians

Consider upper Hessenberg matrices of the form

$$\mathcal{H}^{n\times n}_{\{\theta_k\}}(P) = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,n} \\ s_1 & h_{2,2} & h_{2,3} & \cdots & h_{2,n} \\ 0 & s_2 & h_{3,3} & \cdots & h_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & h_{n,n} \end{bmatrix},$$

with $s_k = e^{i\theta_k}$ (usually $s_k \in \{-1, +1\}$; we do not allow zero s_k entries) and $h_{i,j} \in P$ for $1 \le i \le j \le n$.

For a population P such that $0 \in P$, let $\mathcal{Z}^{n\times n}_{\{\theta_k\}}(P)$ be the subset of $\mathcal{H}^{n}_{\{\theta_k\}}(P)$ where the main diagonal entries are fixed at 0.

Upper Hessenberg Toeplitz Bohemians

Consider upper Hessenberg matrices with a Toeplitz structure of the form

$$\mathcal{M}_{\{\theta_k\}}^{n\times n}(P) = \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_n \\ s & t_1 & t_2 & \cdots & t_{n-1} \\ 0 & s & t_1 & \cdots & t_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s & t_1 \end{bmatrix},$$

with $s = e^{i\theta_k}$ and $t_i \in P$ for $i \leq n$.

Motivation and Applications

Why Bohemians?

- Software testing: we have found bugs in major packages
- Connections to several mathematical topics including
  - Combinatorics
  - Number theory
  - Random matrix theory
  - Graph theory

Why upper Hessenberg?

- Connection to Mandelbrot matrices (which are upper Hessenberg with $P \in \{-1, 0\}$ and the subdiagonal entries are fixed at -1), which gave rise to a new kind of companion matrix[1,2].

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[1] Chan & Corless (2017). A new kind of companion matrix. *Electronic Journal of Linear Algebra*, 32, 335—342.

[2] Chan & Corless (2018). Minimal height companion matrices for Euclid Polynomials. *Mathematics in Computer Science*, 13, 41–56.

What has been done so far?

Upper Hessenberg and Toeplitz Bohemians have been studied quite extensively already. Some questions that have already been solved include

- # matrices
- # singular matrices
- # normal matrices
- # nilpotent matrices
- # stable matrices[3]
- # neutral matrices[4]
- # characteristic polynomials
- max characteristic height[5], # such
- # neutral polynomials[6]
- # stable polynomials
- # distinct k-fold eigenvalues
- # distinct real eigenvalues
- # positive (semi)-definite

---

[3] *Type 1 stable matrix*: a matrix with all of its eigenvalues strictly in the left half plane, *Type 2 stable matrix*: a matrix with all of its eigenvalues inside the unit circle.

[4] Matrices where eigenvalues have zero real part

[5] Largest absolute value of any coefficient of the characteristic polynomial

[6] Characteristic polynomials where all roots have zero real part

Questions that have not been answered (yet)

- # distinct singular values
- maximum departure from normality
- # well-conditioned matrices
- # idempotent matrices
- # rhapsodic matrices[7]
- # orbits of commuting matrices
- # distinct minimal polynomials
- eigenvalue conditioning
- # matrices with only simple eigenvalues

[7] Matrices in which their inverse are also Bohemian

Questions that have not been answered (yet)

- **# distinct singular values**
- maximum departure from normality
- # well-conditioned matrices
- # idempotent matrices
- # rhapsodic matrices[7]
- # orbits of commuting matrices
- # distinct minimal polynomials
- **eigenvalue conditioning**
- **# matrices with only simple eigenvalues**

[7] Matrices in which their inverse are also Bohemian

Bohemians we will be investigating in this talk

1. $\mathcal{H}_{\{0\}}^{n\times n}(\{-1,0,+1\})$
   - Upper Hessenberg matrix with population $\{-1,0,+1\}$ and subdiagonal entries fixed at $e^{0i}=1$
2. $\mathcal{H}_{\{0\}}^{n\times n}(\{0,+1\})$
   - Upper Hessenberg matrix with population $\{0,+1\}$ and subdiagonal entries fixed at 1
3. $\mathcal{H}_{\{0\}}^{n\times n}(\{-1,+1\})$
   - Upper Hessenberg matrix with population $\{-1,+1\}$ and subdiagonal entries fixed at 1
4. $\mathcal{Z}_{\{0\}}^{n\times n}(\{-1,0,+1\})$
   - Upper Hessenberg matrix, where the main diagonal entries are fixed at 0, with population $\{-1,0,+1\}$ and subdiagonal entries fixed at 1
5. $\mathcal{M}_{\{0\}}^{n\times n}(\{-1,0,+1\})$
   - Upper Hessenberg Toeplitz matrix with population $\{-1,0,+1\}$ and subdiagonal entries fixed at 1

Number of Distinct Singular Values

Singular Value Decomposition

Every $m \times n$ matrix $A \in \mathbb{C}^{m \times n}$ may be factored as

$$A = U \Sigma V^H$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices and Σ is an $m \times n$ nonnegative diagonal matrix such that $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$ with $p = \min(m, n)$. The σs are known as singular values, and the factoring is known as the SVD.

The singular values can also be computed by taking the square root of the eigenvalues of $A^H A$.

Finding # of Distinct Singular Values

- We used MATLAB Symbolic Toolbox to calculate the number of distinct singular values
  - To take advantage of its parallel computing toolbox
  - We have yet to use MATLAB Symbolic Toolbox for Bohemians (we have mainly been using MAPLE)
- We computed the number of distinct singular values using two methods:
  - To speed up computation, we used GCD-free factoring; this allows us to compute the number of distinct singular values without needing to actually compute the singular values.
  - Used `svd` function to calculate the singular values symbolically and used the `unique` function (for an array containing all singular values) to determine the number of distinct singular values.
    - We are interested in testing how well MATLAB is able to recognize the equality of two symbolic expressions.

Finding # of Distinct Singular Values

Sample MATLAB Code

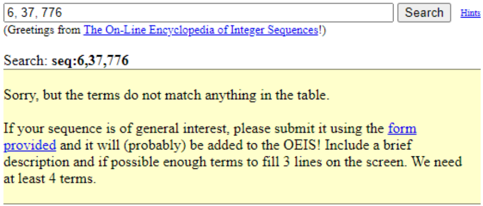
```
>> x = sym('x');
>> A = sym([-1, -1, -1, -1;
            1, -1, -1, -1;
            0, 1, 0, 0;
            0, 0, 1, -1])

A =

[-1, -1, -1, -1]
[ 1, -1, -1, -1]
[ 0,  1,  0,  0]
[ 0,  0,  1, -1]

>> CP = charpoly(A'*A, x)

CP =

x^4 - 11*x^3 + 36*x^2 - 44*x + 16
```

```
>> F = factor(CP)

F =

[x^2 - 7*x + 4, x - 2, x - 2]

>> unique(F)

ans =

[x - 2, x^2 - 7*x + 4]
```

of Distinct Singular Values

Example 1: $\mathcal{H}_{\{0\}}^{n\times n}(\{-1,0,+1\})$

n	#matrices	#distinct σ	MaxOccur	σ_{MaxOccur}
2	27	6	12	$\sqrt{2}$, $\sqrt{\frac{3\pm\sqrt{5}}{2}}$
3	729	37	228	1
4	59,049	776	14,307	0
5	14,348,907	?	?	?


6, 37, 776 Search Hints

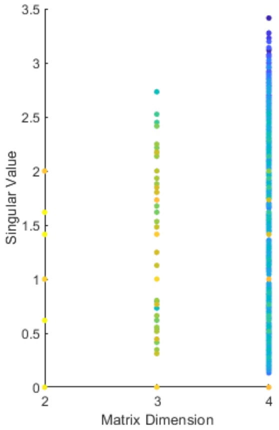
(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,37,776**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

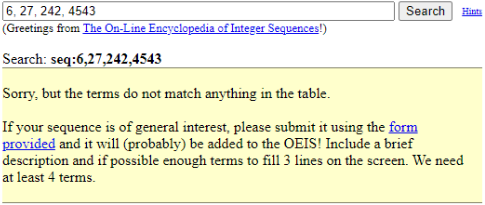
Distribution of σ



of Distinct Singular Values

Example 2: $\mathcal{H}^{n\times n}_{\{0\}}(\{0,+1\})$

n	#matrices	#distinct σ	MaxOccur	σ_{MaxOccur}
2	8	6	4	0
3	64	27	37	1
4	1,024	242	575	1
5	32,768	4543	16,284	1
6	2,097,152	?	?	?


6, 27, 242, 4543 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,27,242,4543**

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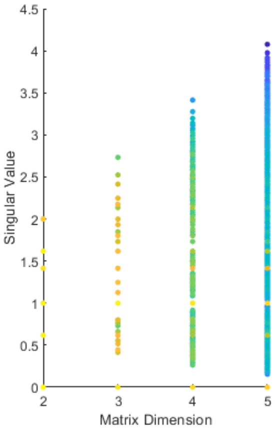
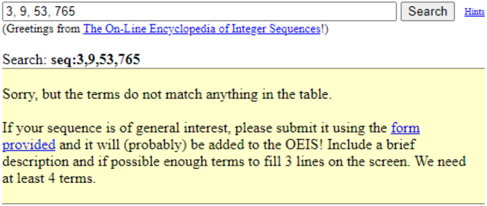
Distribution of σ



of Distinct Singular Values

Example 3: $\mathcal{H}_{\{0\}}^{n\times n}(\{-1,+1\})$

n	#matrices	#distinct σ	MaxOccur	σ_{MaxOccur}
2	8	3	8	$\sqrt{2}$
3	64	9	32	$\sqrt{2}$ and 3 other σs
4	1,024	53	320	$\sqrt{2}$
5	32,768	765	7,648	0
6	2,097,152	?	?	?



3, 9, 53, 765 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:3,9,53,765**

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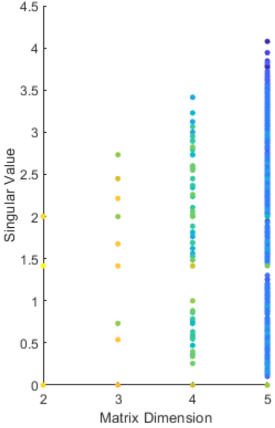
Distribution of σ



of Distinct Singular Values

Example 4: $\mathcal{Z}_{\{0\}}^{n\times n}(\{-1,0,+1\})$

n	#matrices	#distinct σ	MaxOccur	σ_{MaxOccur}
2	3	2	5	1
3	27	8	20	1
4	729	73	393	1
5	59,049	1,899	25,506	1
6	14,348,907	?	?	?

2, 8, 72, 1899 Search Hints

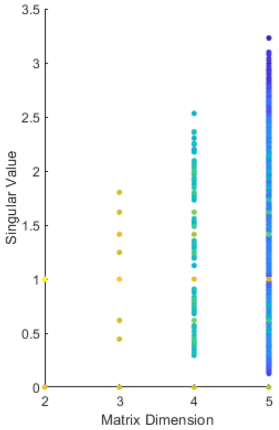
(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:2,8,72,1899**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

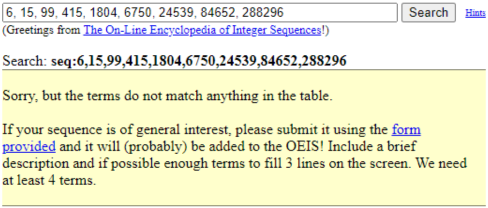
Distribution of σ



$\#$ of Distinct Singular Values

Example 5: $\mathcal{M}^{n\times n}_{\{0\}}(\{-1,0,+1\})$

n	#matrices	#distinct σ	MaxOccur	σ_{MaxOccur}
2	9	6	5	1
3	27	15	16	1
4	81	99	37	1
5	243	415	90	1
6	729	1,804	207	1
7	2,187	6,750	502	1
8	6,561	24,539	1,079	1
9	19,683	84,652	2,366	1
10	59,049	288,296	4,837	1


6, 15, 99, 415, 1804, 6750, 24539, 84652, 288296 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,15,99,415,1804,6750,24539,84652,288296**

Sorry, but the terms do not match anything in the table.

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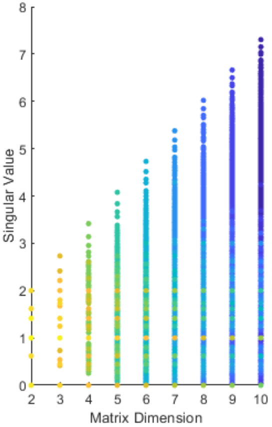
Distribution of σ



Eigenvalue Conditioning

Condition number of a simple eigenvalue

Suppose that $Ax = \lambda x$, where x is a right eigenvector. Also suppose that $y^H A = \lambda y^H$, where y is a left eigenvector. Then, the absolute condition number for a simple eigenvalue is

$$\kappa_{\text{abs}} = \frac{1}{|y^H x|} \, .$$

Eigenvalue Conditioning

We hope to address:

- Number and proportion of matrices with simple eigenvalues
  - Similar to computing the number of distinct singular values, we used GCD-free factoring to determine whether a matrix only contains simple eigenvalues.
- Number of matrices associated with each condition number
- Maximum condition number for the set of shared eigenvalues
  - Since several matrices can contain the same eigenvalues, we want to see out of the set of eigenvalues, what the maximum condition number is.
- Which eigenvalues have the greatest condition number

Number of Matrices with Simple Eigenvalues

Example 1: $\mathcal{H}_{\{0\}}^{n\times n}(\{-1, 0, +1\})$

n	#matrices	#matrices$_{\text{simple }\lambda}$	Proportion
2	27	22	0.8148
3	729	626	0.8587
4	59,049	53,350	0.9035
5	14,348,907	?	?

22, 626, 53350 [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:22,626,53350**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Frequency of Condition Number of Eigenvalue

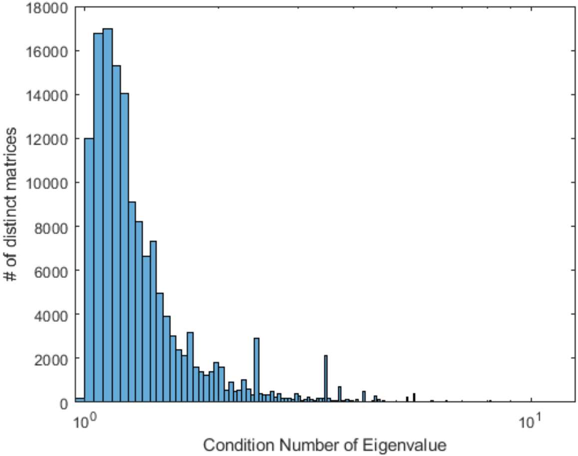


Figure: Histogram of the number of matrices from the Bohemian family $\mathcal{H}_{\{0\}}^{4\times 4}(\{-1,0,+1\})$ with the condition number of eigenvalue.

Eigenvalue Conditioning

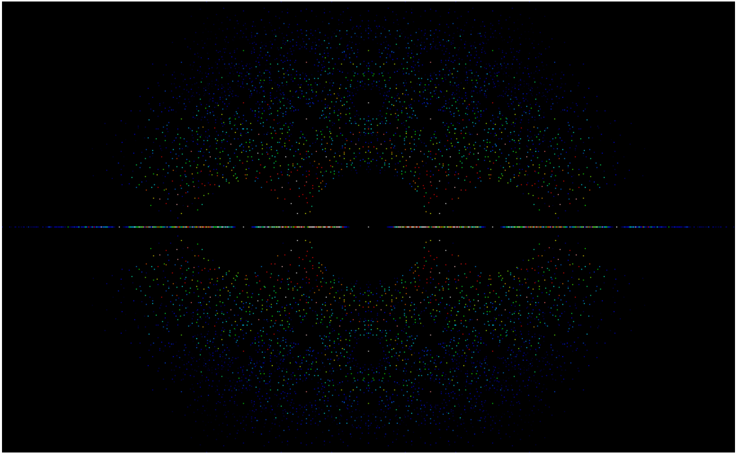


Figure: All eigenvalues of matrices in the subset $\mathcal{H}^{4\times4}_{\{0\}}(\{-1,0,+1\})$ that only contain simple eigenvalues. Eigenvalues are coloured according to maximum condition number of shared simple eigenvalues.

Number of Matrices with Simple Eigenvalues

Example 2: $\mathcal{H}_{\{0\}}^{n\times n}(\{0,+1\})$

n	#matrices	#matrices$_{\text{simple } \lambda}$	Proportion
2	8	6	0.75
3	64	48	0.75
4	1,024	853	0.8330
5	32,768	28,110	0.8578
6	2,097,152	?	?

6, 48, 853, 28110 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,48,853,28110**

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If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Frequency of Condition Number of Eigenvalue

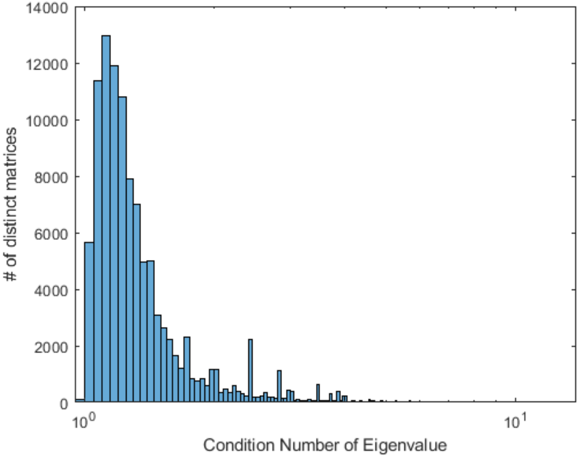


Figure: Histogram of the number of matrices from the Bohemian family $\mathcal{H}^{5\times 5}_{\{0\}}(\{0,+1\})$ with the condition number of eigenvalue.

Eigenvalue Conditioning

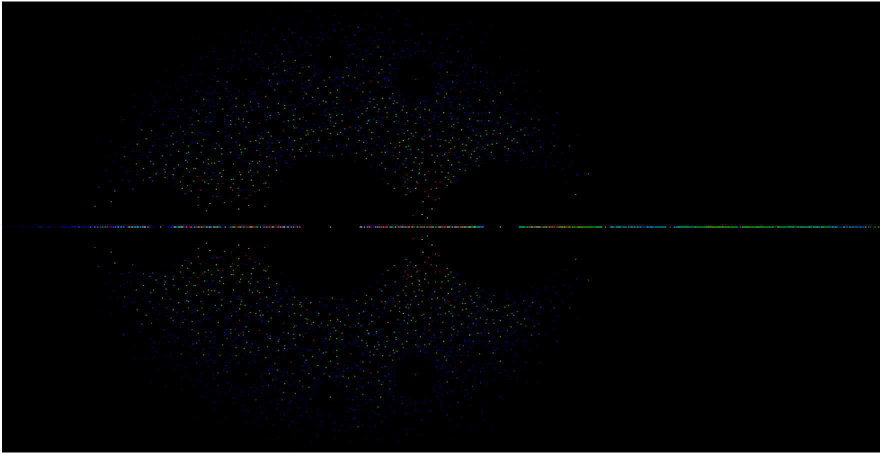
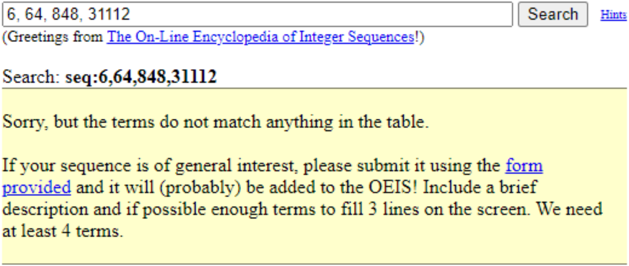


Figure: All eigenvalues of matrices in the subset $\mathcal{H}^{5\times5}_{\{0\}}(\{0,+1\})$ that only contain simple eigenvalues. Eigenvalues are coloured according to maximum condition number of shared simple eigenvalues.

Number of Matrices with Simple Eigenvalues

Example 3: $\mathcal{H}^{n\times n}_{\{0\}}(\{-1,+1\})$

n	#matrices	#matrices$_{\text{simple }\lambda}$	Proportion
2	8	6	0.75
3	64	64	1.0
4	1,024	848	0.8281
5	32,768	31,112	0.9495
6	2,097,152	?	?


6, 64, 848, 31112 Search Hints
(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,64,848,31112**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Frequency of Condition Number of Eigenvalue

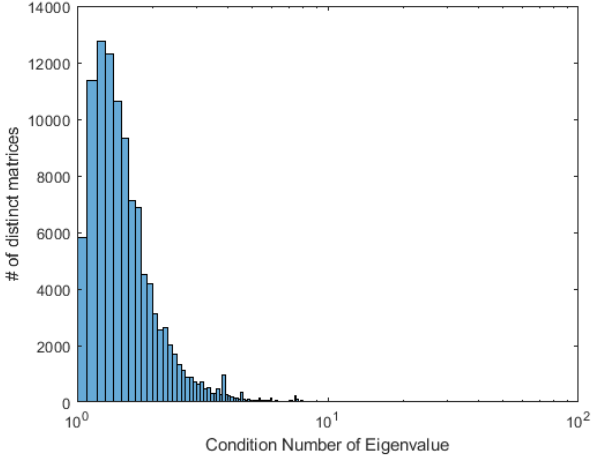


Figure: Histogram of the number of matrices from the Bohemian family $\mathcal{H}^{5\times5}_{\{0\}}(\{-1,+1\})$ with the condition number of eigenvalue.

Eigenvalue Conditioning

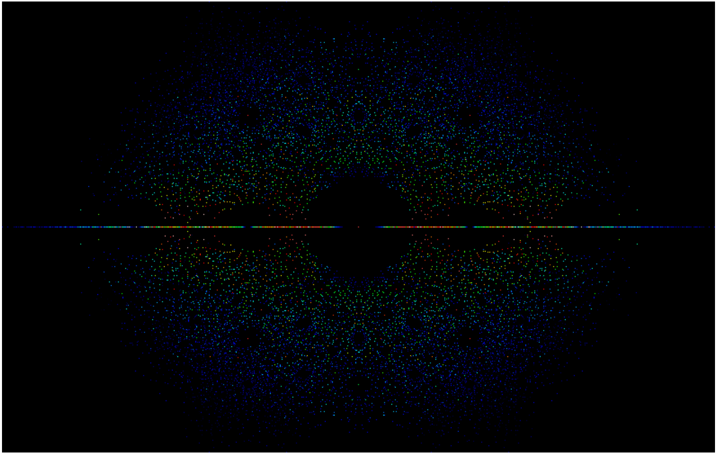


Figure: All eigenvalues of matrices in the subset $\mathcal{H}^{5\times5}_{\{0\}}(\{-1,+1\})$ that only contain simple eigenvalues. Eigenvalues are coloured according to maximum condition number of shared simple eigenvalues.

Number of Matrices with Simple Eigenvalues

Example 4: $\mathcal{Z}_{\{0\}}^{n\times n}(\{-1,0,+1\})$

n	#matrices	#matrices$_{\text{simple }\lambda}$	Proportion
2	3	2	0.6667
3	27	24	0.8889
4	729	626	0.8587
5	59,049	53,386	0.9041
6	14,348,907	?	?

2, 24, 626, 53386 [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:2,24,626,53386**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Frequency of Condition Number of Eigenvalue

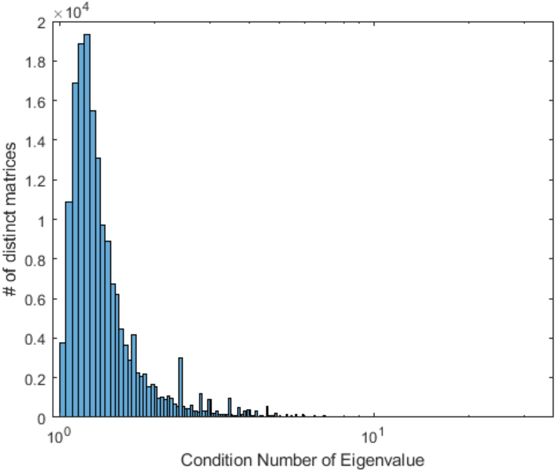


Figure: Histogram of the number of matrices from the Bohemian family $\mathcal{Z}^{5\times5}_{\{0\}}(\{-1,0,+1\})$ with the condition number of eigenvalue.

Eigenvalue Conditioning

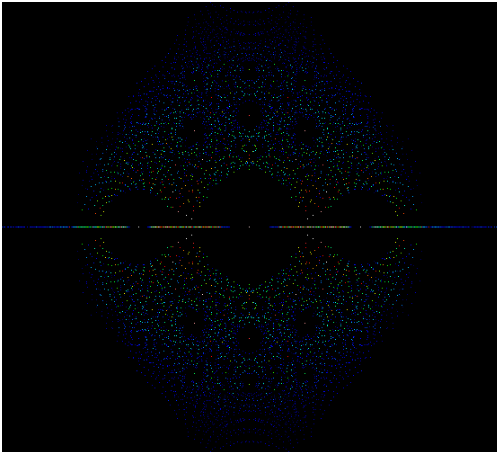


Figure: All eigenvalues of matrices in the subset $\mathcal{Z}_{\{0\}}^{5\times 5}(\{-1, 0, +1\})$ that only contain simple eigenvalues. Eigenvalues are coloured according to maximum condition number of shared simple eigenvalues.

Number of Matrices with Simple Eigenvalues

Example 5: $\mathcal{M}_{\{0\}}^{n\times n}(\{-1,0,+1\})$

n	#matrices	#matrices$_{\text{simple }\lambda}$	Proportion
2	9	6	0.6667
3	27	24	0.8889
4	81	66	0.8148
5	243	228	0.9383
6	729	684	0.9383
7	2,187	2,142	0.9794
8	6,561	6,384	0.9730
9	19,683	19,458	0.9886
10	59,049	58,230	0.9861
11	177,147	176,412	0.9959

6, 24, 66, 228, 684, 2142, 6384, 19458, 58230 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,24,66,228,684,2142,6384,19458,58230**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Frequency of Condition Number of Eigenvalue

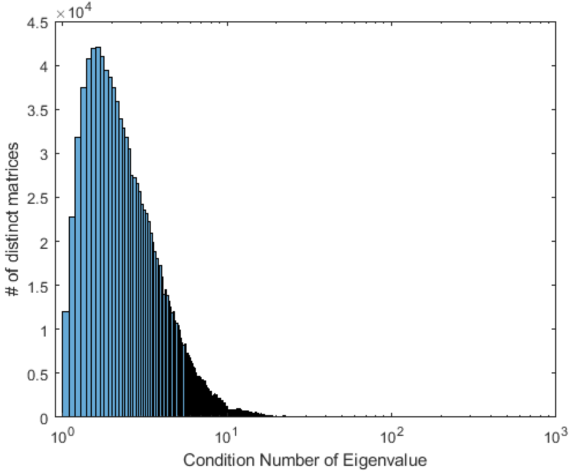


Figure: Histogram of the number of matrices from the Bohemian family $\mathcal{M}_{\{0\}}^{11\times 11}(\{-1,0,+1\})$ with the condition number of eigenvalue.

Eigenvalue Conditioning

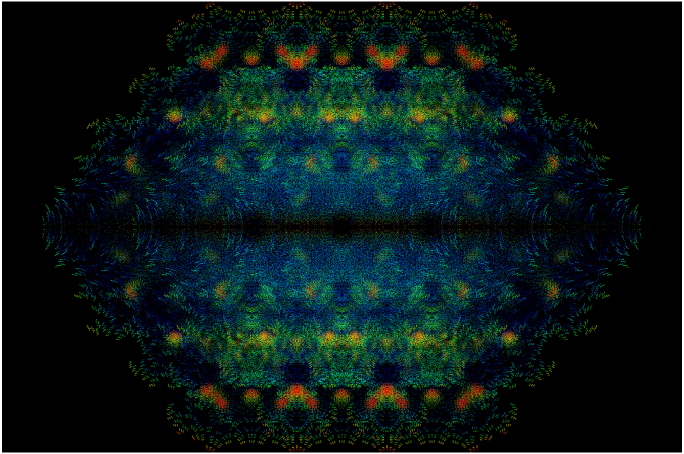


Figure: All eigenvalues of matrices in the subset $\mathcal{M}^{11\times 11}_{\{0\}}(\{-1, 0, +1\})$ that only contain simple eigenvalues. Eigenvalues are coloured according to maximum condition number of shared simple eigenvalues.

Concluding Remarks and Future Works

The class of upper Hessenberg Bohemians gives a useful way to study Bohemians in general.

- Because these classes are simpler than the general case, we were able to compute all results by using brute-force computation.

Future Works:

- Computing higher dimensions for these 5 examples
- Understanding why certain eigenvalues are more sensitive to pertubations than others.
- Understanding the distribution of # of (distinct) matrices corresponding to condition number of the eigenvalues
- Explore other questions such as
  - # well-conditioned matrices
  - # rhapsodic matrices

Thanks for listening!