Reduced Precision Elementary Functions

Jean-Michel Muller
CNRS/LIP, ENS Lyon
France

SIAM 2021 Conference on Computational Science and Engineering

Use of reduced-precision elementary functions ?

Sometimes a very fast rough approximation is useful:

- first step of a more accurate, iterative, computation;
- computer vision, multimedia:
  - tolerate a slight loss in accuracy,
  - need real-time processing;
  - frequently deal with inherently inaccurate data;
- activation functions (sigmoids, tanh...) in neural nets;
- entertainment:
  - Super Mario's pizza does not need to follow the laws of physics accurately,
  - fluidity matters.

Small number formats: the blessing of exhaustivity

- easy testing (try all possible inputs);
- correct rounding becomes easily achievable

  → reproducible results;

  [Low precision *and* very accurate!]
- tabulation becomes an option;
- the return of hw implementation of a kernel of functions?
  - was the case a long time ago (Intel 8087 and followers);
  - Hw faster than Sftw,
  - shortcoming: impossible to fix or improve an already shipped circuit...

    partly vanishes with exhaustive testing and correct rounding.

Methods for elementary function evaluation

Initial range reduction to some interval I.

For input variables $\in I$,

1. polynomial or rational approximations (software); or
2. table-based methods (hardware, software); or
3. shift-and-add algorithms (hardware);

Methods for elementary function evaluation

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For input variables $\in I$,

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2. table-based methods (hardware, software); or
3. shift-and-add algorithms (hardware);
4. *low precision only:* bit-manipulation techniques.

Reduced precision and polynomial or rational approximations

Less accuracy constraints $\rightarrow$ larger domains and/or smaller degrees.

- simpler, or even non-necessary range reduction when uniform approximations can be used.

  Example [Girones et al., 2013, Computer vision]:

  $$k = \frac{153}{256}, \quad \mu = \frac{201}{128},$$
  $$\arctan(x) \approx \mathrm{sign}(x) \cdot \mu \cdot \frac{k \cdot |x| + x^2}{2k \cdot |x| + x^2 + 1}, \text{with error } < 0.0029.$$

  No branchings $\rightarrow$ vector implementations much easier.

- small degrees of polynomial or rational approximations
  $\rightarrow$ fine tuning of approximations,
  $\rightarrow$ exhaustive search of “best” evaluation schemes,
  - in terms of speed (optimal use of arithmetic pipeline)
  - or in terms of evaluation error bound.

Tabulating functions (hardware)

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With current technology:

- small tables: implemented as *boolean functions* of the entries (specific optimizations);
- big tables: all values are stored (matrix of 0s and 1s). Address decoding + propagation of signal through a line of the matrix.

$\rightarrow$ big gap. Threshold at around 2^{10}–2^{12} elements.

The Bipartite-Table Method

- harware-oriented. Goal: overcoming the "2^{12}-element limit"
- Matula & Das Sarma (1995): reciprocal tables (seeds for NR)

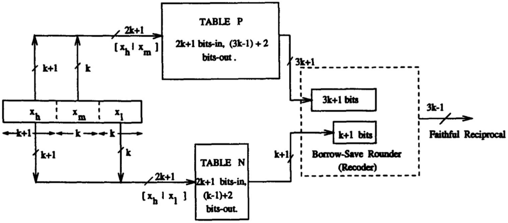


Figure 1: A j+2=3k+1 bits-in j=3k-1 bits out Faithful Bipartite Reciprocal table

- arbitrary (regular enough) functions: Schulte, Stine, (1997).

The Bipartite-Table Method

Approximate $f(x)$, where x is a p-bit fixed-point number in $[0,1]$;

- split x into three k-bit numbers x_0, x_1, and x_2, $k = \lceil p/3 \rceil$:

$$x = x_0 + 2^{-k}x_1 + 2^{-2k}x_2,$$

  with x_i multiple of 2^{-k} and $0 \leq x_i < 1$.

- approximate $f(x)$ by $A(x_0, x_1) + B(x_0, x_2)$, where

$$\begin{cases} A(x_0, x_1) & = & f(x_0 + 2^{-k}x_1 + 2^{-2k-1}) \\ B(x_0, x_2) & = & 2^{-2k}\left(x_2 - \frac{1}{2}\right) \cdot f'(x_0). \end{cases}$$

- 2 tables of $2p/3$ address bits instead of one with p address bits.

A small example

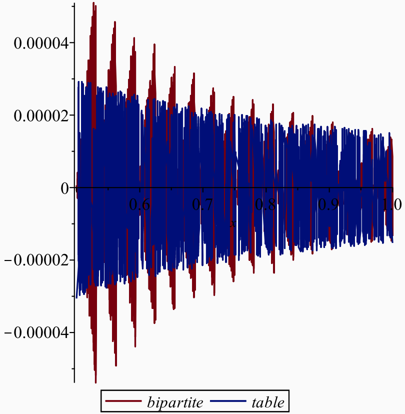
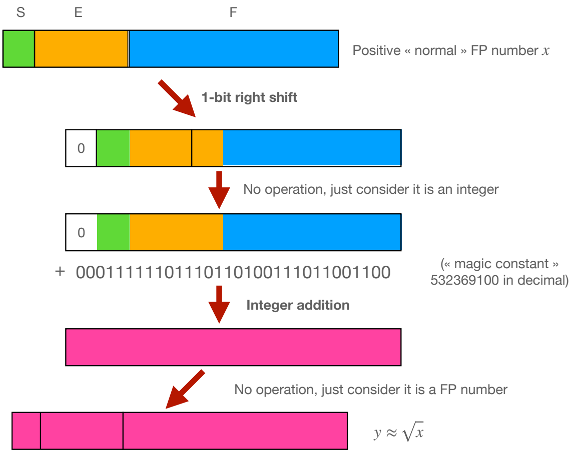


Figure 1: Error of a bipartite approximation of $\ln(x)$ in $[\frac{1}{2}, 1]$ with $k = 5$, compared with error of a 15-address-bit table (table size 16 $\times$ larger).

A very odd trick (game Quake III, 1999)



A similar trick first appears in
The game Quake III Arena

Bit-manipulation techniques

- use the fact that the exponent field of x encodes $\lfloor \log_2 |x| \rfloor$.
- Binary32 (a.k.a. single precision) representation of x:

S_x	E_x	F_x
31	30 … 23	22 … 0

  - 1-bit sign S_x, 8-bit biased exponent E_x, 23-bit fraction F_x s.t.

$$x = (-1)^{S_x} \cdot 2^{E_x - 127} \cdot \left(1 + 2^{-23} \cdot F_x\right).$$

  - the same bit-chain, if interpreted as 2's complement integer, represents the number

$$I_x = (1 - 2S_x) \cdot 2^{31} + \left(2^{23} \cdot E_x + F_x\right).$$

Bit-manipulation method for $\sqrt{x}$

Remember:

$$x = (-1)^{S_x} \cdot 2^{E_x - 127} \cdot \left(1 + 2^{-23} \cdot F_x\right) = (-1)^{S_x} \cdot 2^{e_x} \cdot (1 + f_x).$$

S_x	E_x	F_x
31	30 … 23	22 … 0

- If $e_x = E_x - 127$ is even (i.e., E_x is odd), we use:

$$\sqrt{(1 + f_x) \cdot 2^{e_x}} \approx \left(1 + \frac{f_x}{2}\right) \cdot 2^{e_x/2}, \tag{1}$$

- if e_x is odd (i.e., E_x is even), we use:

$$\begin{aligned} \sqrt{(1 + f_x) \cdot 2^{e_x}} &= \sqrt{4 + \epsilon_x} \cdot 2^{\frac{e_x - 1}{2}} \\ &\approx \left(2 + \frac{\epsilon_x}{4}\right) \cdot 2^{\frac{e_x - 1}{2}} \\ &= \left(\frac{3}{2} + \frac{f_x}{2}\right) \cdot 2^{\frac{e_x - 1}{2}}, \end{aligned} \tag{2}$$

(Taylor series for $\sqrt{4 + \epsilon_x}$ at $\epsilon_x = 0$, with $\epsilon_x = 2f_x - 2$)

Bit-manipulation method for $\sqrt{x}$ (Blinn)

$$x = (-1)^{S_x} \cdot 2^{E_x - 127} \cdot \left(1 + 2^{-23} \cdot F_x\right) = (-1)^{S_x} \cdot 2^{e_x} \cdot (1 + f_x).$$

S_x	E_x	F_x
31	30 … 23	22 … 0

- E_x odd $\rightarrow \left(1 + \frac{f_x}{2}\right) \cdot 2^{\frac{e_x}{2}}$,

$$(1 + F_y \cdot 2^{-23}) \cdot 2^{E_y - 127} \approx (1 + F_x \cdot 2^{-24}) \cdot 2^{\frac{E_x - 127}{2}}$$
$$\Rightarrow E_y = \frac{E_x + 127}{2} \text{ and } F_y = \left\lfloor \frac{F_x}{2} \right\rfloor$$

- E_x even $\rightarrow \left(\frac{3}{2} + \frac{f_x}{2}\right) \cdot 2^{\frac{e_x - 1}{2}}$.

$$(1 + F_y \cdot 2^{-23}) \cdot 2^{E_y - 127} \approx \left(\tfrac{3}{2} + F_x \cdot 2^{-24}\right) \cdot 2^{\frac{E_x - 128}{2}}$$
$$\Rightarrow E_y = \frac{E_x + 127}{2} - \frac{1}{2} \text{ and } F_y = 2^{22} + \left\lfloor \frac{F_x}{2} \right\rfloor$$

In both cases:

$$I_y = \left\lfloor \frac{I_x}{2} \right\rfloor + 127 \cdot 2^{22}$$

Bit-manipulation method for $\sqrt{x}$ (Blinn)

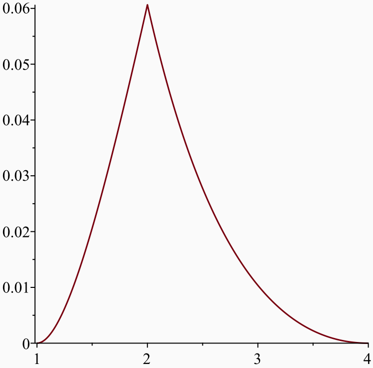


Figure 2: Plot of $(\text{approx} - \sqrt{x})/\sqrt{x}$.

- fast but rough approximation;
- always $\geq \sqrt{x}$ $\rightarrow$ replace $127 \cdot 2^{22}$ by a smaller value?

$\sqrt{x}$ with a better constant

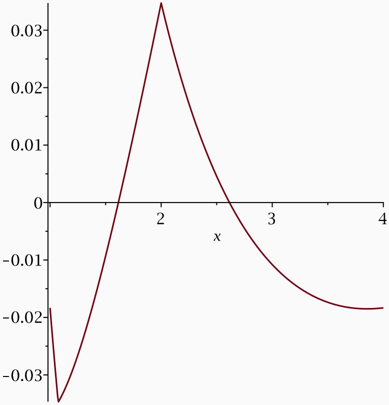


Figure 3: Plot of $(\text{approx} - \sqrt{x})/\sqrt{x}$ with $127 \cdot 2^{22}$ replaced by 532369100.

Quake III function (1999): inverse square root

- $1/\sqrt{x}$, possibly followed by Newton-Raphson iteration;
- reasoning very similar to above-presented $\sqrt{x}$.

$$I_y = -\left\lfloor \frac{I_x}{2} \right\rfloor + 1597463007$$

- non-optimal constant: 1597465647 is slightly better.

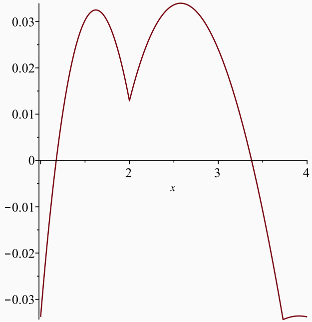


Figure 4: Plot of (approx $- 1/\sqrt{x}) \times \sqrt{x}$ with the "magic constant" 1597463007.

Bit-manipulation technique for the logarithm (Blinn)

$$\begin{aligned} 2^{-23}(I_x - I_1) &= (E_x - 127) + 2^{-23} \cdot F_x \\ &= e_x + f_x \\ &\approx \log_2(x). \end{aligned}$$

$\rightarrow \log_2(x) \approx 2^{-23} \cdot \mathtt{Float}(I_x - I_1).$

where $\mathtt{Float}(I)$ is the FP number mathematically equal to I.

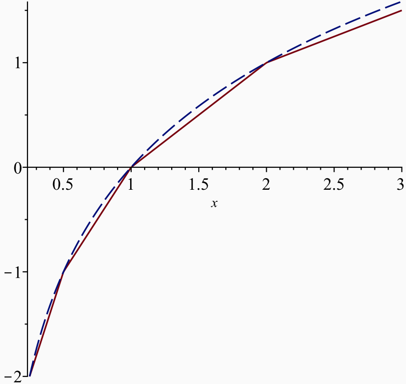


Figure 5: Approx. to $\log_2(x)$

Linear interpolation at powers of 2.

And the winner is. . .

And the winner is. . .

Nobody!

Different targets: hw vs sftw, low accuracy only vs "scalable", only a few functions vs versatile.

- Software:
  - if error $<$ a few % suffices, bit-manipulation techniques hard to beat (typically 1 integer addition and 1 shift). They don't scale up and are not versatile;
  - otherwise: polynomial/rational (scalable and versatile).
- Hardware:
  - bipartite methods very versatile, do not scale-up well;
  - shift-and-add algorithms fine tuning of speed-vs-accuracy compromise. Scale-up well. Moderately versatile.

Thank you!