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Stochastic Rounding and its Probabilistic Backward Error Analysis

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Collaborators

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Stochastic rounding and its probabilistic backward error analysis,
SIAM J. Sci. Comput. 43(1), A566–A585, 2021.

Preliminaries

$$\mathrm{fl}(x \text{ op } y) = (x \text{ op } y)(1+\delta), \quad |\delta| \le u, \text{ op } \in \{+,-,\times,/\}$$

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Lemma (Higham (2002))

If $|\delta_i| \le u$ for $i = 1:n$, and $nu < 1$, then

$$\prod_{i=1}^{n}(1+\delta_i) = 1+\theta_n, \quad |\theta_n| \le \gamma_n,$$

with

$$\gamma_n := \frac{nu}{1-nu} = nu + O(u^2).$$

A probabilistic bound

$$\widetilde{\gamma}_n(\lambda) := \exp\left(\frac{\lambda\sqrt{n}u + nu^2}{1-u}\right) - 1 = \lambda\sqrt{n}u + O(u^2).$$

Theorem (Higham & Mary (2019))

Let $\delta_1, \delta_2, \ldots, \delta_n$ be independent random variables of mean zero with $|\delta_i| \leq u$, $i = 1 : n$. Then for any $\lambda > 0$ we have

$$\prod_{i=1}^{n}(1+\delta_i) = 1 + \theta_n, \quad |\theta_n| \leq \widetilde{\gamma}_n(\lambda)$$

which holds with probability at least

$$P(\lambda) = 1 - 2\exp(-\lambda^2/2).$$

Stochastic rounding

Given adjacent floating-point numbers a, b and $x \in \mathbb{R}$ so that $a \leq x \leq b$, we have

$$\mathrm{fl}(x) = \begin{cases} b \text{ with probability } p = (x-a)/(b-a), \\ a \text{ with probability } 1-p. \end{cases}$$

- Called Mode 1 stochastic rounding (SR).
- Gaining interest in machine learning.

Rounding errors

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Theorem (C, Higham & Mary, 2021)

The rounding errors $\delta_1, \delta_2, \ldots, \delta_n$ produced by stochastic rounding are mean independent, mean zero random variables such that

$$\mathbb{E}(\delta_k) = \mathbb{E}(\delta_k \mid \delta_{k-1}, \ldots, \delta_1) = 0.$$

A new theorem

Theorem (C, Higham & Mary, 2021)

Let $\delta_1, \delta_2, \ldots, \delta_n$ be ***mean independent*** *random variables of mean zero with $|\delta_i| \leq u$, $i = 1 : n$. Then for any $\lambda > 0$ we have*

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- SR satisfies these assumptions (with the substitution $u \leftarrow 2u$).
- Rule of thumb becomes a rule!

Example: inner product

- Want to compute $y = a^T b$, $a, b \in \mathbb{R}^n$.
- When using SR, we have the backward error result:

$$\widehat{y} = (a + \Delta a)^T b,$$
$$|\Delta a| \le \widetilde{\gamma}_n(\lambda)|a| \approx \lambda\sqrt{n}u|a|.$$

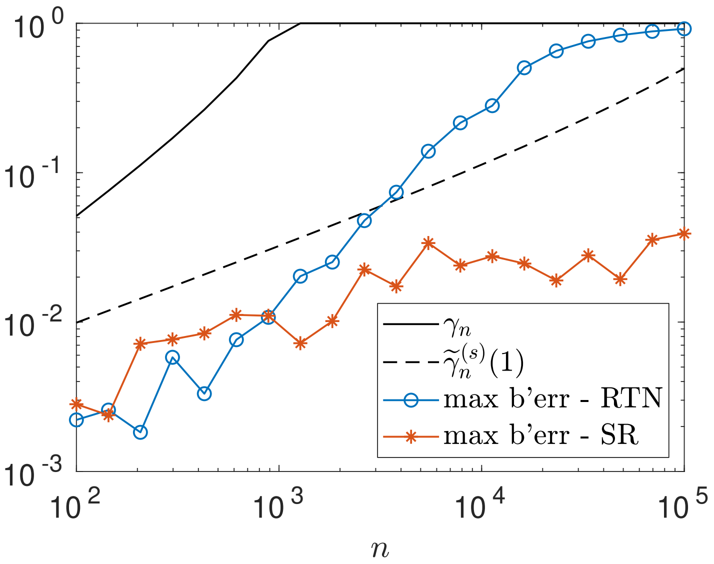
- The result holds with probability at least $1 - 2n\exp(-\lambda^2/2)$.
- Compare with the worst case bound for round to nearest (RTN)

$$|\Delta a| \le \gamma_n|a| \approx nu|a|.$$

Numerical experiments

- Compute inner product $y = a^T b$ for a, b sampled uniformly from $[0, 1]$.
- Work in fp16 ($u = 2^{-11}$).
- Use the implementation of SR provided by `chop` (Higham and Pranesh, 2019).
- **`https://github.com/higham/chop`**

Numerical experiments



Stagnation

- As the intermediate value $y_i = y_{i-1} + a_i b_i$ grows, the spacing between nearby floating-point values increases.
- We reach a point where under RTN the sum can no longer grow.
- SR solves this issue by jumping in the "wrong" direction.

Conclusions

- SR produces mean independent, mean zero rounding errors.
- SR provides backward error bounds that are proportional to $\sqrt{n}u$.
- SR can prove much more accurate than RTN in certain scenarios.

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